151. Commuting Dilations of Self-adjoint Operators

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All operators considered are bounded self-adjoint. Given an operator A on the Hilbert space $\mathfrak{H}$, an operator T on a superspace $\supseteq\mathfrak{H}$ is called a *dilation* of A in case $Af=PTf$ for $f\in\mathfrak{H}$, where P is the projection onto $\mathfrak{H}$. A family $\mathfrak{A}$ of operators on $\mathfrak{H}$ is said to be of $\langle\alpha, \beta\rangle$ *type* in case there is a commutative family $\mathfrak{B}$ of operators on a superspace such that spectra of every member of $\mathfrak{B}$ are contained in the closed interval $[\alpha, \beta]$ and every member of $\mathfrak{A}$ finds a dilation in $\mathfrak{B}$. In the above definition the superspace is not fixed throughout, but depends on $\mathfrak{A}$. In this note an intrinsic description of being of $\langle\alpha, \beta\rangle$ type is given and some of related problems are discussed.

Since under a homothety $A\to\rho A+\xi I$, I being the identity operator, the dilation type $\langle\alpha, \beta\rangle$ changes to $\langle\rho\alpha+\xi, \rho\beta+\xi\rangle$ or $\langle\rho\beta+\xi, \rho\alpha+\xi\rangle$ according as ρ is positive or not, most of discussions can be reduced to the cases of positive (i.e., non-negative definite) operators.

A finite family $\{A_1, \cdots, A_n\}$ of positive operators is said to be γ-*decomposable* in case there is a finite family of positive operators, admitting possible multiplicity, such that the total sum is γI and every A_j is a sum of a suitable subfamily. The definition can be also stated in this way: there is a positive operator-valued, finitely additive measure, with total measure γI, on a Boolean algebra, whose range contains all A's. A family of positive operators is said to be γ-decomposable in case every finite subfamily is γ-decomposable.

Given a 1-decomposable family $\mathfrak{A}=\{A_\lambda : \lambda\in\Lambda\}$, consider the free Boolean algebra with Λ as the set of generators. By the 1-decomposability, for any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of indices there is a normalized, positive operator valued, finitely additive measure on the subalgebra generated by $\lambda_1, \cdots, \lambda_n$, which assigns each A_{λ_j} to λ_j. Since the subalgebra is homomorphic image of the whole algebra (see [2, p. 141]), the measure can be extended over the latter. Now standard arguments based on the weak compactness of the set of positive contractions show that $\mathfrak{A}$ is contained in the range of a normalized, positive operator valued, finitely additive measure on the Boolean algebra. Then the famous theorem of Naimark ([1], [3]) guarantees that a 1-decomposable family admits a commutative family of dilations, consisting of projections, so that it is of $\langle 0, 1\rangle$ type.