# 150. On Indefinite (E. R.)-Integrals. II

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§3. Now, let us prove the following main theorem.

**Theorem.** If $f(x)$ is $\mathcal{D}$-integrable in $I_0=[a, b]$, there exists a measure $\nu$ such that $f(x)$ has a indefinite (E.R. $\nu$)-integral, (E.R. $\nu$) $\int_a^x f(t)dt$, and (E.R. $\nu$) $\int_a^x f(t)dt=(\mathcal{D})\int_a^x f(t)dt$ for all $x\in I_0$.

**Proof.** We may clearly assume that $f(x)=0$ for all $x\in C(I_0)$. If the function $f(x)$ is summable on $I_0$, we have (E.R. $\nu$) $\int_a^x f(t)dt=\int_a^x f(t)dt=(\mathcal{D})\int_a^x f(t)dt$ for every measure $\nu$ which fulfils condition 1*) and 2*) [1].

Next, we shall consider the case in which $f(x)$ is not summable. Let $f(x)$ be a function which is $\mathcal{D}$-integrable but not summable on $I_0$. Then, there exists, by the lemma, a non-decreasing sequence of closed sets $\{F_l\}$ such that (i) $\bigcup_{l=1}^{\infty} F_l=I_0$, (ii) $f(x)$ is summable on $F_l$,

$$\text{(iii)}\quad \left| F(I)-\int_{F_l\cap I} f(x)dx \right| \leq 2^{-l} \text{ for every interval } I\subset I_0, \tag{1}$$

$$\text{(iv)}\quad \sum_{j=1}^{\infty} | F(J_l^j) | \leq 2^{-l} \tag{2}$$

for the sequence of intervals $\{J_l^j\}$ contiguous to the closed set which consists of all points of $F_n$ and end points of $I_0$.

Since $f(x)$ is by hypothesis, not summable, we may assume that

$$\int_{F_l-F_{l-1}} | f(x) | \,dx \geq 2^{-l} \qquad l=1, 2, 3 \cdots$$

$$\text{(we regard } F_0 \text{ as empty).} \tag{3}$$

On account of this and summability of $f(x)$ on $F_l$, we find, for every $l$, a measurable set $H_l\subset F_l$ such that $f(x)\geq f(x')$ for every $x\in H_l$ and $x'\in F_l-H_l$, and

$$\int_{H_l} | f(x) | \,dx=2^{-l}. \tag{4}$$

Writing $\delta_l=\text{mes}\, H_l$, we see at once that

$$\text{mes}\,(F_l-F_{l-1})>\delta_l, \tag{5}$$

$$\delta_l>\delta_{l+1}, \tag{6}$$

$$\text{mes}\,(E)<\delta_l \text{ implies } \int_E | f(x) | \,dx\leq 2^{-l} \tag{7}$$

for every measurable set $E\subset F_l$.

Let $h_l$ and $k_l$ be integers such that

$$(h_l-1)\delta_l< \text{mes}\,(F_l-F_l-1)<h_l\delta_l, \tag{8}$$

$$2^{k_l-1}\delta_{l+1}<\delta_l<2^{k_l}\delta_{l+1}\,. \tag{9}$$