147. Boolean Elements in Lukasiewicz Algebras. II

By Roberto CIGNOLI and Antonio MONTEIRO
Instituto de Matemática Universidad Nacional del Sur,
Bahía Blanca, Argentina

(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

0. *INTRODUCTION.* R. Cignoli has proved the following result:
0.1. *THEOREM: Let A be a Kleene algebra. It is possible to define on A a structure of Lukasiewicz algebra if and only if the family B of all Boolean elements of A has the following properties:*

B 1) B is separating.

B 2) B is lower relatively complete.

The purpose of this note is to show that if, instead of a Kleene algebra, A is a distributive lattice with first (0) and last element (1), then we can define on A a structure of Lukasiewicz algebra if and only if the family B has the properties B 1), B 2), and

B 3) B is upper relatively complete.

We shall use the notations and definitions of [1].

In § 1 we introduce an alternative definition of Lukasiewicz algebra which is useful for the purpose of this paper.

1. *DEFINITION OF LUKASIEWICZ ALGEBRAS.* We can define the notion of (three-valued) Lukasiewicz algebra introduced and developed by Gr. Moisil [3], [4], [5] in the following way [6], [7]:
1.1. *DEFINITION: A (three-valued) Lukasiewicz algebra is a system $(A, 1, \wedge, \vee, \sim, \nabla)$ where $(A, 1, \wedge, \vee, \sim)$ is a de Morgan lattice and ∇ is a unary operator defined on A satisfying the following axioms:*

L 1) $\sim x \vee \nabla x = 1$, L 2) $x \wedge \sim x = \sim x \wedge \nabla x$,

L 3) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

In [6] (Theorem 4.3) it was proved that in a (three-valued) Lukasiewicz algebra the operation $\sim$ also satisfies the condition

K) $x \wedge \sim x \leq y \vee \sim y$,

that is, the system $(A, 1, \wedge, \vee, \sim)$ is not only a de Morgan algebra but a Kleene algebra.

A. Monterio has proved that if we postulate the condition K), then we can replace axiom L 3) of definition 1.1 by the weaker

$L'3)$ $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$.

More exactly:
1.2. *THEOREM: Let $(A, 1, \wedge, \vee, \sim, \nabla)$ be a system such that $(A, 1, \wedge, \vee, \sim)$ is a Kleene algebra and ∇ is a unary operator defined on A satisfying axioms L 1), L 2), and $L'3)$. Then* $(A, 1, \wedge,$