146. Boolean Elements in Lukasiewicz Algebras. I

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0. *INTRODUCTION*. In the theory of the (three-valued) Lukasiewicz algebras founded by Gr. Moisil, the possibility operator plays an important role. Moisil denotes the operator by M and we shall denote by ∇ it defined on a distributive lattice A and it is uniquely determined by the set K of all elements $k\in A$ such that $\nabla k=k$.

The purpose of this note is to establish characteristic properties of the family K. In § 1 we summarize some theorems on closure operators defined on lattices. In § 2, we study these operators in the case of Kleene algebras, and in § 3 we apply these results to the problem suggested by A. Monteiro.*)

1. *CLOSURE LATTICES*. Let $(L, 0, 1, \wedge, \vee)$ be a lattice with first and last elements. If a unary operator ∇ is defined on L such that:

$C\,1)$ $\nabla 0=0$, $\qquad$ $C\,2)$ $x\leq \nabla x$,

$C\,3)$ $\nabla(x\vee y)=\nabla x\vee \nabla y$, $\qquad$ $C\,4)$ $\nabla\nabla x=\nabla x$,

we shall say that the system $(L, 0, 1, \wedge, \vee, \nabla)$ is a *closure lattice*, and the operator ∇ is a *closure operator*. This notion is a generalization of closure operators on topological spaces and was studied by N. Nakamura [17] (see also [16] and [18]).

It is easy to prove that:

$C\,5)$ If $x\leq y$, then $\nabla x\leq \nabla y$, or equivalently,

$C\,6)$ $\nabla(x\wedge y)\leq \nabla x\wedge \nabla y$.

In [18] it was proved that

1.1. *The family K of all invariant elements of a closure operator has the following properties*:

$K\,1)$ *K is a sub-lattice of L containing* 0 *and* 1.

$K\,2)$ *K is lower relatively complete*: *that is, for all $x\in L$, the set $\{k\in K : x\leq k\}$ has an infimum belonging to K.*

Moreover we have

$$(1) \qquad \nabla x=\wedge\{k\in K : x\leq k\}.$$

Conversely, if K is a subset of L with the properties $K\,1)$ *and* $K\,2)$, (1) *defines a closure operator ∇ on L, and K is the set of all invariant elements by ∇.*

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