141. Some Mapping Theorems for the Numerical Range

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The purpose of the present note is to prove some mapping theorems for the numerical range of a linear operator, somewhat analogous to the spectral mapping theorem. Because of the peculiarity that the numerical range is always convex, the theorems are rather restricted in validity compared with the spectral mapping theorem.

In what follows we mean by an operator A a bounded linear operator in a Hilbert space H with domain H. The numerical range and the spectrum of A are denoted by $W(A)$ and $S(A)$, respectively. It is well known that $S(A)\subset\overline{W(A)}$ ($\overline{\ \ }$ denotes the closure) and that $\overline{W(A)}$ is the closed convex hull of $S(A)$ if A is normal.

Also we need the notion of the *convex kernel* K of a non-empty set E in the complex plane; K is the set of all points z such that E is star-shaped relative to z. It is known[1] that K is a convex subset of E, $K=E$ if E is convex, and that K is compact if E is.

Theorem 1. Let $f(z)$ be a rational function with $f(\infty)=\infty$. Let E' be a compact convex set in the complex plane, let $E=f^{-1}(E')$ and let K be the convex kernel of E. If A is an operator with $W(A)\subset K$, then $W(f(A))\subset E'$.

Remark 2. Under the assumptions of the theorem, E', E, and K are all compact and f has no poles in E. Since $S(A)\subset\overline{W(A)}\subset K\subset E$, $f(A)$ is well defined. K may be empty, in which case the theorem is of no use. For K to be non-empty, it is necessary that E be connected and contain all critical points of f(so that E' contain all branch points of the inverse function f^{-1}).

Corollary 3.[2] If $W(A)$ is a subset of the closed unit disk, the same is true of $W(A^n)$, $n=2, 3, \cdots$.

For the proof of Theorem 1 and other theorems given below, we use the following lemma, the proof of which is trivial. We set $Re\,A=(A+A^*)/2$, $Im\,A=(A-A^*)/2i$, and note that $([Re\,A]u, u)=Re(Au, u)$, $([Im\,A]u, u)=Im(Au, u)$ for any $u\in H$.

Lemma 4. Let A be a nonsingular operator. Then $Re\,A\geqq 0$ is

1) See [2] and [5].

2) This theorem is due to C. A. Berger [1]. The author was told that it was also proved by C. M. Pearcy. For $n=2^m$ it had been proved earlier by H. Fujita (unpublished).