140. On Lacunary Fourier Series

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Our first theorem is as follows:

Theorem 1. If the function f has the Fourier series

$$(1)\qquad f(x)\sim\sum_{k=1}^{\infty}(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx)$$

where

$$(2)\qquad n_{k+1}-n_k>An_k^\beta\ (A\text{ constant and }0<\beta\leqq 1)$$

and if f satisfies the α-Lipschitz condition ($\alpha>0$) at a point x_0, that is,

$$|f(x_0+t)-f(x_0)|\leqq A|t|^\alpha\quad\text{as}\quad t\to 0,$$

then we have

$$a_{n_k}=O(1/n_k^{\alpha\beta}),\quad b_{n_k}=O(1/n_k^{\alpha\beta})\qquad (k=1,2,\cdots).$$

This is a generalization of theorems of Kennedy [1] and Tomić [2].

Proof. a) The case $1>\alpha>0$. We can suppose that $x_0=0$. Let c_{n_k} be the n_k-th complex Fourier coefficient of f, then

$$c_{n_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-in_kx}dx.$$

We can suppose that[1)]

$$(2')\qquad n_{k+1}-n_k\geqq An_k^\beta\quad\text{and}\quad n_k-n_{k-1}\geqq An_k^\beta$$

and then we have

$$c_{n_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)T_{M_k}(x)e^{-in_kx}dx$$

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1) If $\beta=1$, that is, $n_{k+1}/n_k\geqq\lambda>1$, then we can take $A=(\lambda-1)/\lambda$. In the case $0<\beta<1$, we can suppose that $n_{k+1}\geqq 2n_k$. For, if not, that is, if $n_{k+1}-n_k\geqq A'n_k^\beta$ for a constant A' and $n_{k+1}>2n_k$, then we insert the term $c_{n_{k'}}e^{in_{k'}x}$ with $n_{k'}=n_k+A'n_k^\beta$, then

$$n_{k'}-n_k=A'n_k^\beta,\quad n_{k+1}-n_{k'}=(n_{k+1}-n_k)-A'n_k^\beta\geqq n_k-A'n_k^\beta\geqq A'n_k^\beta$$

for large k. If, further, $n_{k+1}>2n_{k'}$, then we insert also the term $c_{n_{k''}}e^{in_{k''}x}$ with $n_{k''}=n_{k'}+A'(n_{k'})^\beta$. Thus proceeding we get the sequence $(n_k^{(\nu)};\ \nu=1,2,\cdots,j)$ such that

$$n_k<n_{k'}<n_{k''}<\cdots<n_k^{(j)}<n_{k+1}$$

and

$$n_{k+1}\leqq 2n_k^{(j)},\quad n_k^{(\nu+1)}\leqq 2n_k^{(\nu)}(\nu=1,2,\cdots,j-1),\ n_{k'}\leqq 2n_k,$$
$$n_k^{(\nu+1)}-n_k^{(\nu)}\geqq A'(n_k^{(\nu)})^\beta\ (\nu=1,2,\cdots,j-1),\ n_{k+1}-n_k^{(j)}\geqq A'(n_k^{(j)})^\beta,\ n_{k'}-n_k\geqq A'n_k^\beta.$$

This procedure is possible for all sufficiently large k. Now, instead of f, consider the function $g(x)=f(x)+h(x)$ where $h(x)\sim\sum_{\nu,k}c_k^{(\nu)}e^{in_k^{(\nu)}x}=\sum d_ke^{im_kx}$. We can take $(c_k^{(\nu)})$ such that h is sufficiently smooth. Then g satisfies the condition of f and the Fourier exponents (m_k) of g satisfy (2′) with $A=A'/2^\beta$.