139. A Perturbation Theorem for Semi-groups of Linear Operators

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Let A be the infinitesimal generator of a contraction semi-group T_t of class (C_0) on the Banach space X.[1] A is thus a closed linear operator with the domain $D(A)$ and the range $R(A)$ both in X such that: i) $D(A)$ is dense in X, and ii) the resolvent $(\lambda I-A)^{-1}$ exists as a bounded linear operator on X into X satisfying the estimate $\|\lambda(\lambda I-A)^{-1}\|\leqq 1$ for all $\lambda>0$. Let B likewise be the infinitesimal generator of another contraction semi-group of class (C_0) on X. Then the condition $D(B)\geqq D(A)$ implies, by the closed graph theorem, that there exist positive constants a and b such that

$$\|Bx\|\leqq a\|Ax\|+b\|x\| \quad \text{for} \quad x\in D(A).$$

As an important remark to Theorem 2 in H. F. Trotter [1] (Cf. T. Kato [1]), E. Nelson [1] proved that $(A+B)$ with the domain $D(A+B)=D(A)$ is the infinitesimal generator of a contraction semi-group of class (C_0) if we can take $a<1/2$.

The purpose of the present note is to propose a sufficient condition in order that Nelson's hypothesis be satisfied. We shall prove

Theorem. Let $0<\alpha<1$. Let $\hat{A}_\alpha=-(-A)^\alpha$ be the fractional power of A, and let us assume that $D(B)\geqq D(\hat{A}_\alpha)$. Then $(A+B)$ with the domain $D(A+B)=D(A)$ is the infinitesimal generator of a contraction semi-group of class (C_0) on X.

Corollary. Assume, furthermore, that A generates a holomorphic semi-group, then $(A+B)$ with the domain $D(A+B)=D(A)$ also generates a holomorphic semi-group.

Remark 1.[2] The fractional power $\hat{A}_\alpha$ is defined as the infinitesimal generator of the semi-group of class (C_0):

$$(1) \qquad \hat{T}_{t,\alpha}x=\int_0^\infty f_{t,\alpha}(s)T_s x\,ds \quad (t>0,\ x\in X),$$

where

$$(2) \qquad f_{t,\alpha}(s)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^\alpha}dz(\sigma>0,\ t>0,\ s\geqq 0),$$

the branch of z^α being taken so that $Re(z^\alpha)>0$ for $Re(z)>0$. According to V. Balakrishnan [1], we have $D(\hat{A}_\alpha)\geqq D(A)$ and

$$(3) \qquad (-A)^\alpha x=\frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{\alpha-1}(\lambda I-A)^{-1}(-Ax)d\lambda \quad \text{for } x\in D(A).$$

1) See, e.g., E, Hille-R. S. Phillips [1] or K. Yosida [1].
2) See, e.g., K. Yosida [1].