204. Decompositions of Generalized Algebras. II

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Theorem 3. *Every genalgebra $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ with finitary operations is isomorphic with a subdirect product of subdirectly irreducible genalgebras.*

Proof. Consider arbitrary elements $x, y \in G$, $a, b \in A$ such that $x \neq y$ and $a \neq b$. Let $\mathcal{L}(x, y; a, b)$ be the family of all reduced congruences (θ, φ) of $\mathcal{L}(x, y; a, b)$ such that

$$(x, y) \notin \theta \quad \text{and} \quad (a, b) \notin \varphi.$$

Since $(\Delta_G, \Delta_A) \in \mathcal{L}(x, y; a, b)$, then $\mathcal{L}(x, y; a, b) \neq \varnothing$. It is partially ordered and every linearly ordered subset of it possesses an upper bound given by its join. Hence, by Zorn's lemma, $\mathcal{L}(x, y; a, b)$ has a maximal element $(\theta_{xy}, \varphi_{ab})$. To show that the quotient genalgebra

$$\mathfrak{S}/(\theta_{xy}, \varphi_{ab})=\langle G/\theta_{xy}, o_1, \cdots, o_n, A/\varphi_{ab}\rangle$$

is subdirectly irreducible, it suffices to show that it has no proper reduced congruences and hence no proper congruences. If it does possess proper reduced congruences, let $(\tilde{\theta}_\lambda, \tilde{\varphi}_\lambda)$ $(\lambda \in \Lambda)$ be the family of all reduced congruences in $\mathfrak{S}/(\theta_{xy}, \varphi_{ab})$. By Theorem C each such congruence $(\tilde{\theta}_\lambda, \tilde{\varphi}_\lambda)$ corresponds to a reduced congruence $(\theta_\lambda, \varphi_\lambda)$ in $\mathfrak{S}$ such that

$$(\theta_\lambda, \varphi_\lambda) \gneqq (\theta_{xy}, \varphi_{ab}).$$

Clearly, $\theta_\lambda \supsetneqq \theta_{xy}$ for all $\lambda \in \Lambda$; for, if $\theta_\lambda = \theta_{xy}$, then $\varphi_\lambda = \varphi_{ab}$, since both congruences are reduced. Thus we have $\bigcap_{\lambda \in \Lambda} \theta_\lambda \supsetneqq \theta_{xy}$ and in any case

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi_\lambda) \gneqq (\theta_{xy}, \varphi_{ab}).$$

The reduction

$$\bigcap_{\lambda \in \Lambda} (\theta_\lambda, \varphi)$$

of the congruence on the left side must properly contain the congruence on the right side; for, if $\varphi \subsetneqq \varphi_{ab}$, then

$$(\bigcap_{\lambda \in \Lambda} \theta_\lambda, \varphi) \cap (\theta_{xy}, \varphi_{ab}) = (\bigcap_{\lambda \in \Lambda} \theta_\lambda \cap \theta_{xy}, \varphi \cap \varphi_{xy}) = (\theta_{xy}, \varphi)$$

contrary to the fact that $(\theta_{xy}, \varphi_{ab})$ is reduced. Whence the genalgebra $\mathfrak{S}/(\theta_{xy}, \varphi_{ab})$ is subdirectly irreducible. Obviously,

$$\bigcap_{x \neq y} \bigcap_{a \neq b} (\theta_{xy}, \varphi_{ab}) = (\bigcap_{x \neq y} \theta_{xy}, \bigcap_{a \neq b} \varphi_{ab}) = (\Delta_G, \Delta_A)$$

and therefore the final conclusion follows.

Theorem 4. *The necessary and sufficient conditions for a genalgebra $\mathfrak{S}=\langle G, o_1, \cdots, o_n, A\rangle$ to be isomorphic to a direct product of genalgebras $\mathfrak{S}_\lambda=\langle G_\lambda, o_1^\lambda, \cdots, o_n^\lambda, A_\lambda\rangle (\lambda \in \Lambda)$ are that* (1) *there exists*