200. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XVIII

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Theorem 48. Let $\chi(\lambda)$ and σ be the same notations as before; and let $\hat{T}(\rho)$ denote the definite integral $\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt$. Then $\hat{T}(\rho)$ is not only a monotone decreasing function of ρ but also a convex function of $\log\rho$ in the interval $\sigma<\rho<\infty$.

Proof. Let c be any finite value (inclusive of zero); let $\hat{n}(\rho, c)$ denote the number of c-points, with due count of multiplicity, of $\chi(\lambda)$ in the domain $D_\rho\{\lambda: \rho<|\lambda|\leqq\infty\}$ with $\sigma<\rho<\infty$; let $\hat{n}(\infty, c)$ denote the number of c-points of $\chi(\lambda)$ at $\lambda=\infty$, that is, let $\hat{n}(\infty, c)$ be k or 0 according as c is 0 or not, on the assumption that the point at infinity is a zero-point of order k of $\chi(\lambda)$; let $C_{-\hat{n}(\infty,c)}$ denote C_{-k} or 1 according as c is 0 or not; and let

$$\hat{N}(\rho, c)=\int_\rho^\infty\frac{\hat{n}(r, c)-\hat{n}(\infty, c)}{r}dr-\hat{n}(\infty, c)\log\rho \qquad (\sigma<\rho<\infty),$$

$$\hat{m}(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[\chi(\rho e^{-it}), c]}dt \qquad (\sigma<\rho<\infty),$$

and

$$P(c)=\log|C_{-\hat{n}(\infty,c)}|-\left[1-\frac{\hat{n}(\infty, c)}{k}\right]\log\sqrt{1+\frac{1}{|c|^2}},$$

where we may and do assume that $\left[1-\frac{\hat{n}(\infty, c)}{k}\right]\log\sqrt{1+\frac{1}{|c|^2}}$ vanishes at $c=0$. Then

$$\hat{N}(\rho, c)+\hat{m}(\rho, c)+P(c)=\begin{cases}N(\rho, c)+m(\rho, c)-m(\infty, c) & (c\neq 0)\\ \tilde{N}(\rho, 0)+\tilde{m}(\rho, 0) & (c=0),\end{cases}$$

where $N(\rho, c)$, $\tilde{N}(\rho, c)$, $m(\rho, c)$, $\tilde{m}(\rho, 0)$, and $m(\infty, c)$ are the same notations as those used in Theorems 46 and 47. Let A be the Riemann sphere, a sphere with unit diameter touching the complex λ-plane at the origin 0, and $d\omega(c)$ an areal element at a unique point on A corresponding to a point c in that λ-plane. Since, as can be found from the geometrical meaning of $[\chi(\rho e^{-it}), c]$,

$$\iint_A\log\frac{1}{[\chi(\rho e^{-it}), c]}d\omega(c)\equiv Q$$

is a positive constant irrespective of ρe^{-it} and χ, it is obvious from