199. Axiom Systems of B-algebra. II

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In the first note [2], we gave axiom systems of B-algebra. A B-algebra $M=\langle x, 0, *, \sim\rangle$ is given by the following axioms:

B 1 $x*y\leqslant x$,

B 2 $(x*z)*(y*z)\leqslant(x*y)*z$,

B 3 $x*y\leqslant(\sim y)*(\sim x)$,

B 4 $0\leqslant x$,

where $x\leqslant y$ means $x*y=0$, and if $x\leqslant y$, $y\leqslant x$, then we write $x=y$. There are some axiom systems which is equivalent to B 1$\sim$$B$ 4. For the details, see [1],[2], and [3].

In this note, we shall show the following

Theorem. A B-algebra $M=\langle X, 0, *, \sim\rangle$ *is characterized* by

L 1 $x*(\sim y)\leqslant x*(z*y)$,

L 2 $x*y\leqslant x*(y*z)$,

L 3 $(x*(y*z))*(x*y)\leqslant x*(\sim z)$,

L 4 $0\leqslant x$.

The conditions L 1$\sim$$L$ 4 are an algebraic formulation of Lukasiewicz axioms of classical propositional calculus.

We first prove $B\Rightarrow L$.

As shown in [1], if $x\leqslant y$ in a B-algebra, then $z*y\leqslant z*x$ for any $z\in X$. Hence, by B 1, we have $x*y\leqslant x*(y*z)$. On the other hand, by (8) in [1], $z*y\leqslant\sim y$. Therefore we have $x*(\sim y)\leqslant x*(z*y)$. Next we have the following relation.

$$(x*(y*z))*(x*y)=(\sim(y*z)*(\sim x))*(\sim y*\sim x)\leqslant(\sim(y*z)*\sim y)*(\sim x)$$
$$=(y*(y*z))*\sim x\leqslant x*\sim(y*(y*z)).$$

On the other hand, by $y*z\leqslant y*z$, we have $y*(y*z)\leqslant z$. Hence $\sim z\leqslant\sim(y*(y*z))$. Therefore we have

$$(x*(y*z))*(x*y)\leqslant x*\sim(y*(y*z))\leqslant x*(\sim z),$$

which completes the proof of $B\Rightarrow L$.

Now we shall prove $L\Rightarrow B$.

From L 1 and L 2, we have

(1) $x\leqslant y*z$ implies $x\leqslant\sim z$ and $x\leqslant y$.

By L 2, we have $(x*y)*x\leqslant(x*y)*(x*(y*z))=0$. Hence

(2) $x*y\leqslant x$,

which is B 1. From L 3, we have

(3) $x\leqslant\sim z$ implies $x*(y*z)\leqslant x*y$.

(4) $x\leqslant\sim z$, $x\leqslant y$ imply $x\leqslant y*z$.

By L 1 and (2), we have