195. A Characterization of Boolean Algebra

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 13, 1965)

In my note [4], I gave an algebraic formulation of propositional calculi. Under the same idea, I shall give a characterization of Boolean algebra. The fundamental axioms are given by the algebraic formulation of E. Mendelson axioms [3].

Let $\langle X, 0, *, \sim\rangle$ be an algebra satisfying the axioms given below, where 0 is an element of a set X, and $*$ is a binary operation and $\sim$ is an unary operation. If $x*y=0$, $x, y\in X$, we write $x\leqslant y$ and $\leqslant$ introduce an order relation on X.

1 $x*y\leqslant x$,

2 $(x*z)*(y*z)\leqslant(x*y)*z$,

3 $x*(y*\sim x)\leqslant(\sim y)*(\sim x)$,

4 $0\leqslant x$,

5 $x\leqslant y$ and $y\leqslant x$ imply $x=y$.

If we define $x\vee y=\sim(\sim y*x)$, $x\wedge y=y*(\sim x)$ *and put* $1=\sim 0$, *then* we shall prove that *the algebra* $M=\langle X, 0, *, \sim\rangle$ *is a Boolean algebra with* 1 *as the unit on the operations* $\vee$, $\wedge$, *and* $\sim$. To prove it, we need some lemmas given in [4]. We do not give the proofs of these lemmas (see [4]).

(1) $0*x=0$.

(2) $x*x=0$.

(3) $x\leqslant y$ and $y\leqslant z$ imply $x\leqslant z$.

(4) $x*y\leqslant z$ implies $x*z\leqslant y$.

(5) $x\leqslant y$ implies $z*y\leqslant z*x$ and $x*z\leqslant y*z$.

(6) $y*x=(y*x)*x$.

The relation of (4) is called the *commutative law*. Further we shall prove some propositions from the axioms and the propositions (1)$\sim$(6) which are proved from the axioms 1 and 2.

(7) $x*(x*(\sim x))=0$, $x*(\sim x)=x$.

Let $y=x$ in axiom 3, then by axiom 2, we have (7) and $x*(\sim x)=x$ by axiom 1.

(8) $x*(\sim y)\leqslant y*(\sim x)$.

From axioms 1 and 3, we have

$$x*(y*(\sim x))\leqslant(\sim y)*(\sim x)\leqslant\sim y,$$

hence by (4), we have $x*(\sim y)\leqslant y*(\sim x)$.

(9) $x*(\sim y)=y*(\sim x)$.

This follows from (8) (see [4]), and consequently M is a BN-algebra. Hence we have $x\leqslant\sim(\sim x)$, and M is an NBN-algebra by