194. On Near-algebras of Mappings on Banach Spaces

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1. A real vector space $\mathcal{A}$ is called *a near-algebra* if, for any pair of elements f and g in $\mathcal{A}$, the product fg is defined and satisfies the following two conditions:

(1) $(fg)h=f(gh)$; (2) $(f+g)h=fh+gh$.

The left distributive law: $h(f+g)=hf+hg$ is not assumed. Therefore, a near-algebra is a near-ring which has firstly been defined by [4, pp. 71–74].

A subset $\boldsymbol{I}$ of a near-algebra $\mathcal{A}$ is called *an ideal* if (1) $\boldsymbol{I}$ is a linear subset of $\mathcal{A}$; (2) $f\in\boldsymbol{I}$, $g\in\mathcal{A}$ imply $fg, gf\in\boldsymbol{I}$.

Let E be a real Banach space. Let f and g are mappings of E into E. We define the linear combination $\alpha f+\beta g$ (α and β are real numbers) by

$$(\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x) \quad \text{for every} \quad x\in E,$$

and the product fg by

$$(fg)(x)=f[g(x)] \qquad \text{for every} \quad x\in E.$$

Let $\mathcal{A}$ be a near-algebra whose elements are mappings of E into E. If $\mathcal{A}$ contains the Banach algebra $\boldsymbol{L}$ of all bounded linear mappings of E into E (the norm of $\boldsymbol{L}$ is $\|l\|=\sup_{\|x\|\leq 1}\|l(x)\|$ for $l\in\boldsymbol{L}$), then, for any ideal $\boldsymbol{I}$ of $\mathcal{A}$, the set

$$\boldsymbol{I}(\boldsymbol{L})=\boldsymbol{I}\cap\boldsymbol{L}$$

is an ideal of the Banach algebra $\boldsymbol{L}$.

Examples. Let $\boldsymbol{B}$ be the near-algebra of all bounded (i.e., transforms every bounded set into a bounded set) and continuous mappings. The following subsets are ideals (cf. [3]).

1. The set $\boldsymbol{I}(E)$ of all constant mappings, in other words, $\boldsymbol{I}(E)$ is the set of all mappings $C_a(a\in E)$ such that $C_a(x)=a$ for every $x\in E$.

2. The set $\boldsymbol{C}$ of all compact (i.e., transfroms every bounded set into a compact set) and continuous mappings.

3. The set $\boldsymbol{EB}$ of all entirely bounded (i.e., transforms the space E into a bounded set) and continuous mappings.

It is obvious that $\boldsymbol{B}$ contains $\boldsymbol{L}$ and

$\boldsymbol{I}(E)\cap\boldsymbol{L}=\boldsymbol{EB}\cap\boldsymbol{L}=0$ (zero-ideal of $\boldsymbol{L}$);

$\boldsymbol{C}\cap\boldsymbol{L}=\boldsymbol{CL}$ (the set of all compact continuous linear mappings on E).

2. A mapping f of E into E is said to be (*Fréchet*) *differ-*