191. On Complete Degrees

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(Comm. by Zyoiti SUETUNA, M.J.A., Dec. 13, 1965)

In his paper [2], R. M. Friedberg proved that a degree of recursive unsolvability $\boldsymbol{a}$ is complete if and only if $\boldsymbol{a} \geqq \boldsymbol{0}'$. The aim of this note is to prove the following: *for each degree* $\boldsymbol{a}$, *there exist infinitely many independent degrees* $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *whose completion are* $\boldsymbol{a}$ *if and only if* $\boldsymbol{a} \geqq \boldsymbol{0}'$. This will be shown as a corollary to the following.

Theorem. *For each degree* $\boldsymbol{a}$, *there exist infinitely many degrees* $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *such that*:

(1) $\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \cdots$ *are independent*,

(2) $\boldsymbol{b}_i' = \boldsymbol{b}_i \cup \boldsymbol{0}' = \boldsymbol{a} \cup \boldsymbol{0}'$ *for* $i = 0, 1, \cdots, n, \cdots$

Let $\alpha(x)$ be a function of degree $\boldsymbol{a}$. We shall construct a function $\lambda x i \beta(x, i)$ such that $\lambda x \beta(x, i) (= \beta_i(x))$ is not recursive in $\lambda x z \beta(x, z + sg((z+1) \dot{-} i)) (= \beta^i(x, z))$ and satisfies (2). And let $\boldsymbol{b}_i$ be the degree of $\beta_i(x)$. As in [1], $\lambda x i \beta(x, i)$ is constructed by defining inductively functions $\psi(s)$ and $\nu(s)$ such that

$$\beta(x, i) = (\psi(s))_{x,i} \quad \text{for each} \quad x < \nu(s) \quad \text{and each} \quad i < \nu(s).$$

1. First, we shall define a recursive predicate $\mathrm{comp}(s_1, s_2)$ and function $\phi(e, v)$ of degree $\boldsymbol{0}'$ as follows:

$$\mathrm{comp}(s_1, s_2) \equiv (u_1)_{u_1 < 1h(s_1)} (u_2)_{u_2 < 1h(s_2)} (u_3)_{u_3 < \min(1h(s_1), 1h(s_2))} [(s_1)_{u_1} \neq 0 \ \& \ (s_2)_{u_2} \neq 0 \ \& \ (s_1)_{u_3} = (s_2)_{u_3}],$$

$$\phi(e, v) = \begin{cases} \mu s (T_1^1(s, e, e) \ \& \ \mathrm{comp}(s, v)) \\ \qquad \text{if } (Es)(T_1^1(s, e, e) \ \& \ \mathrm{comp}(s, v)), \\ 0 \qquad \text{otherwise}, \end{cases}$$

where $T_1^1(\prod_{u<y} p_u^{f(u)+1}, e, x) \equiv T_1^f(e, x, y)$.

Now, we shall define the functions $\nu(s)$ and $\psi(s)$ simultaneously by the induction on the number s, and put $\beta(x, i) = (\psi(s))_{x,i}$ for each $x < \nu(s)$ and each $i < \nu(s)$.

Stage $s = 0$.

$\nu(0) = 0$,

$\psi(0) = 1$.

Stage $s + 1$.

Case 1: $(Ey) T_1^2(\widetilde{\beta}^{(s)0}(y, y), (s)_1, \nu(s), y)$.

This means that

$(Ey)(Eb)[b \neq 0 \ \& \ (i)_{i<y}((b)_i \neq 0) \ \& \ (i)_{i<y}(j)_{j<y}((b)_{i,j} < 2) \ \&$