12. A Duality Theorem for Locally Compact Groups. II

By Nobuhiko TATSUUMA
Department of Mathematics, Kyoto University
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1. Let G be a locally compact group, Ω be the set of all equivalence classes of unitary representations of G. We consider a representative $D=\{U_g^D, \mathfrak{H}^D\}$ of each element in Ω. Denote by $\boldsymbol{T}=\{T(D)\}$ an operator field over Ω, and call $\boldsymbol{T}$ *admissible* when

(1) $T(D)$ is a unitary operator in $\mathfrak{H}^D$ for any D in Ω.

(2) $$U_1(T(D_1)\oplus T(D_2))U_1^{-1}=T(D_3),$$

(3) $$U_2(T(D_1)\otimes T(D_2))U_2^{-1}=T(D_4),$$

for arbitrary unitary equivalence relation U_1(resp. U_2) between $D_1\oplus D_2$(resp. $D_1\otimes D_2$) and D_3(resp. D_4).

In the previous paper [1], we showed,

Proposition. *For any admissible operator field $\boldsymbol{T}$, there exists unique element g in G such that*

$T(D)=U_g^D$, *for any D in Ω.*

The present work is devoted to prove,

Theorem. *The assumption* (1) *about unitarity of $T(D)$ is replaceable by weaker assumption,*

(1′) *For regular representation R of G, $T(R)$ is a non-zero bounded operator in $L^2(G)$, and $T(D)$ is a closed operator in $\mathfrak{H}^D$ for any D in Ω.*

2. Proof of the theorem.

Lemma. *Under the assumption* (1′),

$$\| T(R) \| = 1.$$

In fact, the general theory shows,

$$\| T_1\otimes T_2 \| \leqq \| T_1 \| \| T_2 \|,$$
$$\left\| \sum_\alpha \oplus T_\alpha \right\| = \sup_\alpha \| T_\alpha \|.$$

While as shown in [1], $R\otimes R$ is equivalent to a multiple of R, so the conditions (2) and (3) lead us to

$$\| T(R) \|^2 \geqq \| T(R)\otimes T(R) \| = \| T(R) \|,$$

then $\| T(R) \| \geqq 1$, because of $T(R)\neq 0$. If $\| T(R) \| = a>1$, there exist $\varepsilon>0$ such that $(a-\varepsilon)^2>a$, and a non-zero vector f in $L^2(G)$ such as $\| T(R)f \| > (a-\varepsilon)\| f \|$.

$$\| T(R) \| \| f \|^2 = \| T(R)\otimes T(R) \| \| f\otimes f \| \geqq \| T(R)f\otimes T(R)f \| =$$
$$= \| T(R)f \|^2 > (a-\varepsilon)^2 \| f \|^2 > a \| f \|^2.$$

That contradicts. q.e.d.