8. *Some Theorems in B-algebra*

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In my notes ([1], [2], and [3]), I gave algebraic formulations of classical propositional calculus, and some characterization of Boolean algebra. In this note, I shall give proofs of some results in classical propositional calculus by view of algebraic formulations.

Let $\boldsymbol{M}=\langle X, 0, *, \sim\rangle$ be a B-algebra, i.e., $\boldsymbol{M}$ satisfies the following axioms:

1 $x*y\leqslant x$,

2 $(x*z)*(y*z)\leqslant(x*y)*z$,

3 $x*y\leqslant(\sim y)*(\sim x)$,

4 $0\leqslant x$,

5 $x\leqslant y$ and $y\leqslant x$ imply $x=y$,

where $x\leqslant y$ means $x*y=0$.

Theorem 1. *In a B-algebra* $\boldsymbol{M}$, *we have*

$$(x*u)*(x*z)\leqslant z*(y*x).$$

The formula in theorem 1 is used to give axioms of classical propositional calculus by A. Rose [6]. For proofs, we freely use some results mentioned in my notes [1], [2], and [3]. We refer, for example, the proposition (16) in my note [2] as ((16) in [2]).

Proof.
$$\begin{aligned}(x*u)*(x*z)&=(x*(x*z))*u && ((1)\text{ in }[2])\\ &\leqslant(z*(\sim x))*u && ((13)\text{ in }[3])\\ &=(z*u)*(\sim x) && ((1)\text{ in }[2])\\ &\leqslant z*(\sim x). && ((5)\text{ in }[2])\end{aligned}$$

On the other hand, we have $y*x\leqslant\sim x$ from ((8) in [1]). Hence by ((5) in [2]), we have $z*(\sim x)\leqslant z*(y*x)$. Therefore we have

$$(x*u)*(x*z)\leqslant z*(y*x).$$

The following is an algebraic formulation of a formula given by C. Meredith (see A. N. Prior [5]).

Theorem 2. *In a B-algebra* $\boldsymbol{M}$, *we have*

$$(x*s)*(x*t)\leqslant t*(z*(((\sim s)*(\sim z))*(y*x))).$$

Proof. $(x*s)*(x*t)=((\sim s)*(\sim x)*((\sim t)*(\sim x)\leqslant((\sim s)*(\sim t))*(\sim x)=((\sim s)*(\sim x))*(\sim t)=(x*s)*(\sim t)=t*\sim(x*s)$.

Further, by $y*(x*(\sim y))\leqslant y*x$ in [3] and $x*y\leqslant\sim y$, we have

$$z*((x*s)*(\sim z)\leqslant z*(x*s)\leqslant\sim(x*s).$$

Hence

$$\begin{aligned}t*(\sim(x*s))&\leqslant t*(z*((x*s)*(\sim z)\\ &=t*(z*((\sim s*\sim x)*(\sim y))\\ &=t*(z*((\sim s*\sim z)*(\sim x)).\end{aligned}$$