7. *An Algebra Related with a Propositional Calculus*

By Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

In this note, we shall consider a new algebra induced by the *BCI*-system of propositional calculus by C. A. Meredith quoted into A. N. Prior, Formal Logic ([4], p. 316).[1)]

Unfortunately, we can not find the details on the *BCI* and *BCK*-systems in literatures. For the completeness, giving its detail, we shall develop our consideration.

If we take the *BCI*-system or the weak positive implicational calculus by A. Church, these systems are given by the following axioms.

BCI-system: *CCpqCCqrCpr*, *CpCCpqq*, and *Cpp*,

WPI-system: *CCpCpqCpq*, *CCqrCCpqCpr*, *CCpCqrCqCpr*, and *Cpp*.

In these systems, we can not deduce an important thesis: *CpCqp*. From an attempt of algebraic formulations, we have a quite different situation from our former discussions (see [1], [2]).

Let $\boldsymbol{M}=\langle X, 0, *\rangle$ be an abstract algebra consisting of a set X with an element 0 and a binary operation $*$. If $\boldsymbol{M}$ satisfies the following conditions *BCI* $1\sim5$, it is called a *BCI-algebra*.

BCI 1 $(x*y)*(x*z)\leqslant z*y$,

BCI 2 $x*(x*y)\leqslant y$,

BCI 3 $x\leqslant x$,

BCI 4 $x\leqslant y$, $y\leqslant x$ imply $x=y$,

BCI 5 $x\leqslant 0$ implies $x=0$,

where $x\leqslant y$ means $x*y=0$.

Here we do not assume $0*x=0$, i.e. $0\leqslant x$. This is an essential part and differs from axiom systems formulated in our previous notes [1], [2]. *BCI* 5 shows that $x*0=0$ implies $x=0$. And we have $0*x=0*0=0$. Hence if $x*0=0*x=0$, then $x=0$.

From the first axiom, we have the following important results:

(1) $x\leqslant y$ implies $z*y\leqslant z*x$. $x\leqslant y$, $y\leqslant z$ imply $x\leqslant z$.

By (1), if $x=y$, $y=z$, then $x=z$.

Theorem 1. *The second axiom in BCI-algebra is replaced by*

(2) $(x*y)*z\leqslant(x*z)*y$.

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1) In my seminar on mathematical logic, Mr. Shôtarô Tanaka announced forming rules to produce a single axiom from several axioms of *CN*-types in propositional calculi. His results are based on so called *BCI*, *BCK*-systems introduced by C. A. Meredith (see A. N. Prior [4], p. 316).