6. *Axiom Systems of B-algebra. III*

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In this paper, we shall give an algebraic formulation of the axiom system of propositional calculus given by Lukasiewicz and Tarski (see [1]), and prove that this axiom system is equivalent to a B-algebra defined by K. Iséki (see [2].)

Let $\langle X, 0, *, \sim\rangle$ be an abstract algebra satisfying axioms:

(1) $x*w\leqslant(x*((((u*t)*(s*t)*((u*s)*r))*((\sim t*s)*\sim r)))*((y*z)*y)$.

(2) $0\leqslant x$.

D 1 If $x\leqslant y$ and $y\leqslant x$, then we put $x=y$.

D 2 $x\leqslant y$ means $x*y=0$.

(For details of the notions, see [2].)

In his paper [2], K. Iséki defines the notions of B-algebra $\langle X, 0, *, \sim\rangle$. The axioms are given by the following conditions:

B 1 $x*y\leqslant x$,

B 2 $(x*z)*(y*z)\leqslant(x*y)*z$,

B 3 $x*y\leqslant\sim y*\sim x$,

B 4 $0\leqslant x$,

and D1, D2.

Theorem. *A B-algebra is characterized by axioms* (1) *and* (2).

K. Iséki has proved that the axiom (1) is true in any B-algebra (see [3]). Therefore, we shall prove the converse. The fundamental ideas of the proof is due to my paper [4].

In axiom (1), we substitute z for w, $(x*y)*x$ for x and y, $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for z, $((x*y)*x)*z$ appears in the left side. At the same time, the right side is equal to 0, because it is axiom (1) which is substituted $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for w, $(x*y)*x$ for x, x for y and y for z in axiom (1) respectively. Therefore by (2), D1 and D2, we have

(3) $(x*y)*x\leqslant z$.

In this thesis, put $z=((x*y)*x)*z$, then by (2) and D1, we have $(x*y)*x=0$. Hence by D2, we have

(4) $x*y\leqslant x$.

Let us put $x=((((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r))*((x*y)*x)$, $y=x$, $z=y$, $w=(x*y)*x$ in axiom (1), then the right side is equal to 0, because it is identical with the expression which is substituted $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for x, $(x*y)*x$ for y, $(x*y)*x$ for z in (3). The second and third terms of the left side are equal