5. *On Axiom Systems of Propositional Calculi. XIV*

By Yasuyuki IMAI and Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

All this while, we considered axiom systems of classical propositional calculus. In this note, we shall treat the implicational (propositional) calculus originated by A. Tarski and P. Bernays. The fundamental axioms of the implicational calculus are given by

1 $CpCqp$,

2 $CCpqCCqrCpr$,

3 $CCCpqpp$

and two usual rules of inference.

In his Formal Logic [2], A. N. Prior has proved several theses from the above axioms. We shall follow an algebraic formulation of implicational calculus to prove some theses (for the detail [1]). To do so, we consider an algebra $\boldsymbol{M}=\langle X, 0, *\rangle$ satisfying the following conditions:

$I\,1$ $x*y\leqslant x$,

$I\,2$ $(x*y)*(x*z)\leqslant z*y$,

$I\,3$ $x\leqslant x*(y*x)$,

$I\,4$ $0\leqslant x$,

$I\,5$ $x*y=0$ if and only if $x\leqslant y$.

If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$. The algebra $\boldsymbol{M}$ is called an *I-algebra*.

First we shall show some simple fundamental lemmas.

(1) $0*x=0$.

(2) $x*x=0$, i.e. $x\leqslant x$.

(3) $x*y=0, y*z=0$ imply $x*z=0$, i.e. if $x\leqslant y, y\leqslant z$, then $x\leqslant z$.

(4) $x\leqslant y$ implies $z*y\leqslant z*x$.

To prove (2), put $z=y*z$ in $I\,2$, then we have

$$(x*y)*(x*(y*z))\leqslant(y*z)*y=0.$$

Hence we have

(5) $x*y\leqslant x*(y*z)$.

Next, $y=x, z=y*x$ in (5), then $x*x\leqslant x*(x*(y*x))=0$ by $I\,3$. Hence $x*x=0$, i.e. $x\leqslant x$. By $I\,2$, $z*y=0, x*z=0$ imply $x*y=0$. Hence $x\leqslant z, z\leqslant y$ imply $x\leqslant y$, which means (3). (4) follows from $I\,2$, putting $z*y=0$.

From $I\,1, I\,3$, we have

(6) $x=x*(y*x)$.

By (3), we have $(x*y)*z\leqslant x*y\leqslant x$. Hence

(7) $(x*y)*z\leqslant x$.