1. On Relative Maximal Ideals in Lattices

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1. Introduction. Let S be a sublattice of a lattice L. An ideal M of L shall be called a *relative maximal ideal* with respect to S, like that in a ring, when M is maximal among ideals which are disjoint to S. It was pointed out by Grätzer and Schmidt [1] that there is a close connection between relative maximal ideals and prime ideals. In the present paper we intend to make some additional researches to them and give an assertion analogous to Cohen's theorem in ideal theory for rings.

Again the properties of relative maximal ideals are useful for the decomposition theories in distributive lattices. So we shall give in § 3 new proofs of Kurosch-Ore Theorem concerning the decomposition of elements, which is generalized by Dilworth and Crawley [4], and Hashimoto's theorem [3] concerning the decomposition of ideals.

2. Relative maximal ideals. Let P be a prime ideal of a lattice L, then the complement $L-P$ of P is a dual prime ideal. So every prime ideal P of a lattice L becomes a relative maximal ideal with respect to a sublattice $L-P$. Concerning the converse we shall show the theorem of Grätzer and Schmidt [1] in a somewhat generalized form.

Theorem 1. *Each of the following conditions are necessary and sufficient in order that a lattice L be distributive;*

(1) *every relative maximal ideal of L is prime;*

(2) *every relative maximal ideal of L with respect to a one-element sublattice is prime.*

Proof. Let M be a relative maximal ideal with respect to a sublattice S of a distributive lattice L. Suppose that M is not prime. Then there exist elements x, y such that $x \notin M$, $y \notin M$, and $x \cap y \in M$. $M \cup (x] \supsetneqq M$ and $M \cup (y] \supsetneqq M$ imply $\{M \cup (x]\} \cap S \ni s_1$ and $\{M \cup (y]\} \cap S \ni s_2$ by the maximality of M, hence $\{M \cup (x]\} \cap \{M \cup (y]\} \ni s_1 \cap s_2$. Since the ideals of a distributive lattice themselves form a distributive lattice, $s_1 \cap s_2 \in \{M \cup (x]\} \cap \{M \cup (y]\} = M \cup \{(x] \cap (y]\} = M \cup (x \cap y] = M$, which is a contradiction. Obviously (1) implies (2), accordingly we need only prove that (2) implies the distributivity of L. If a lattice L is not distributive, there exists in L a sublattice isomorphic to the lattice of Fig. 1 or Fig. 2. But in both cases, the relative maximal ideal with respect to b containing the principal