28. Minimal or Smallest Relation of Given Type

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1. This note is to announce the improvement and development of the results in [4] and also to report a brief note of a part of [5]. The earlier paper [4] discussed the smallest of the relations of a given type which contain a given relation, and as its application, the smallest congruence of certain type. In this note we shall treat minimal or smallest relation of given types in the most general cases.

Following Birkhoff [1], we define minimal (smallest) element and maximal (greatest) element in a partially ordered set E with an ordering $\leqq$.

An element a of E is called a minimal (maximal) element of E if

$$x \in E,\ x \leqq a\ (x \geqq a) \text{ imply } x = a.$$

An element a of E is called a smallest (greatest) element of E if

$$x \geqq a\ (x \leqq a) \text{ for all } x \in E.$$

Let S be a set and let $\rho, \sigma \cdots$ denote binary relations on S i.e. subsets of $S \times S$. Let $\mathscr{B}_0$ be the set of all binary relations (which we shall call "relations") on S. $\mathscr{B}_0$ is a complete lattice with respect to inclusion where the empty relation $\square$ is smallest and the universal relation $\omega = S \times S$ is greatest.

2. A subset $\mathscr{T}$ of $\mathscr{B}_0$ is called a "pretype" of relations on S or briefly a pretype on S if $\mathscr{T}$ shall contain $\square$ and if $\mathscr{T}$ is a non-empty subset of $\mathscr{B}_0$; a pretype $\mathscr{T}$ is called a "type" of relations if a pretype $\mathscr{T}$ contains ω. A type $\mathscr{T}$ is called a "basic type" if a type $\mathscr{T}$ satisfies the following condition: for any subset $\{\rho_\xi\}$ of $\mathscr{T}$, the intersection $\bigcap_\xi \rho_\xi \in \mathscr{T}$. A basic type is a complete lattice contained in $\mathscr{T}$ but not necessarily sublattice of $\mathscr{B}_0$. Each relation ρ belonging to a pretype $\mathscr{T}$ is called a $\mathscr{T}$-relation on S.

Let $\mathscr{T}$ be a pretype on S and let σ be a non-empty element of $\mathscr{T}$. An element ρ_σ of $\mathscr{T}$ is called a minimal $\mathscr{T}$-relation containing σ if ρ_σ is a minimal element of the set of all relations $\rho(\in \mathscr{T})$ which contain σ; ρ_σ is called the $\mathscr{T}$-relation generated by σ if ρ_σ is a smallest element of the set of all relations $\rho(\in \mathscr{T})$ which contain σ. An element ρ_0 of $\mathscr{T}$ is called a minimal $\mathscr{T}$-relation on S or we say S has a minimal $\mathscr{T}$-relation if ρ_0 is a minimal element of $\mathscr{T} \backslash \square$ which denotes the set of all non-empty elements of $\mathscr{T}$. An element ρ_0 is called a smallest $\mathscr{T}$-relation on S or we say S has a smallest $\mathscr{T}$-relation if ρ_0 is a smallest element of $\mathscr{T} \backslash \square$.