26. Characterizations of BCI, BCK-Algebras

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In this note, we shall consider some characterizations of the *BCI*, *BCK*-algebras defined in [1]. By a *BCI-algebra*, we mean an algebra $M=\langle X, 0, *\rangle$ with an element 0 and a binary operation $*$ satisfying the following conditions *BCI* 1~5:

BCI 1 $(x*y)*(x*z)\leqslant z*y$,

BCI 2 $x*(x*y)\leqslant y$,

BCI 3 $x\leqslant x$,

BCI 4 $x\leqslant y, y\leqq x$ imply $x=y$.

BCI 5 $x\leqslant 0$ implies $x=0$,

where $x\leqslant y$ means $x*y=0$.

BCI 5 is equivalent to: $x*0=0$ implies $x=0$. If *BCI* 5 is replaced by *BCI* 6: $0\leqslant x$ for every $x\in X$, the algebra M is called *BCK-algebra*.

In [1], we proved that

(6) $(y*x)*(z*x)\leqslant y*z$

holds in the *BCI*-algebra. We first prove the following

Theorem 1. *The BCI-algelra is characterized by BCI* 2~5 *and* (6).

Proof. (6) implies the following results:

(7) If $y\leqslant z$, then $y*x\leqslant z*x$.

(8) If $x\leqslant y$, $y\leqslant z$, then $x\leqslant z$.

By (6) and (7), we have

(9) $((y*x)*(z*x))*u\leqslant (y*z)*u$.

We substitute $y*u$ for z in (9), then by *BCI* 2, we have

(10) $(y*x)*((y*u)*x)\leqslant u$.

In formula (10), let $x=y$, $y=x*z$, and $u=(x*y)*z$, then

$$((x*z)*y)*(((x*z)*((x*y)*z))*y)\leqslant (x*y)*z.$$

The second term of the left side is equal to 0, hence if $x*y\leqslant z$, i.e. $(x*y)*z=0$, then in the formula above, the right side is equal to 0, so we have $x*z\leqslant y$ by *BCI* 5. Therefore, we have

(11) If $x*y\leqslant z$, then $x*z\leqslant y$.

Hence, by (11) and (6),

BCI 1 $(x*y)*(x*z)\leqslant z*y$.

It is obvious from [1] that the converse holds. The proof of Theorem 1 is complete.

By Theorem 1, we have the following