25. *Axiom Systems of B-algebra. V*

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In this note, we shall give a characterization of B-algebra by an algebraic formulation of an axiom system of propositional calculus by B. Sobociński.

We introduce a binary operation $*$ and an unary operation $\sim$ on X. Consider an abstract algebra $\langle X, 0, *, \sim\rangle$ satisfying the following conditions:

B 1　$x*y\leqslant x$,

B 2　$(x*y)*(z*y)\leqslant(x*z)*y$,

B 3　$x*y\leqslant\sim y*\sim x$,

B 4　$0\leqslant x$,

B 5　$x\leqslant y$ and $y\leqslant x$ imply $x=y$,

B 6　$x*y=0$ if and only if $x\leqslant y$

(For details, see [1].)

The algebra $\langle X, 0, *, \sim\rangle$ is called a B-algebra.

By the same idea, we can formulate an axiom system by B. Sobociński as follows:

S 1　$x*y\leqslant\sim y$,

S 2　$(x*y)*z\leqslant x$,

S 3　$(x*(y*z))*(x*y)\leqslant x*\sim z$,

and B 4, B 5, B 6.

In the case of propositional calculus, it is called the (S_1)-system.

We shall prove that two axiom systems are equivalent, therefore any B-algebra, hence a Boolean algebra is characterized by the above axiom S 1–3 and B 4–6.

First we shall show that the axioms of B-algebra are derived from the axioms of (S_1)-system.

In axiom S 3, put $x=(x*y)*z$, $y=x$, then we have $((x*y)*z)*(x*z))*(((x*y)*z)*x)\leqslant((x*y)*z)*\sim z$. In axiom S 1, put $x=x*y$, $y=z$, then we have $((x*y)*z)*\sim z=0$ by B 6. Hence $((x*y)*z)*(x*z)\leqslant((x*y)*z)*x$. The right side is equal to 0 by S 2, then

(1)　$(x*y)*z\leqslant x*z$.

In (1), put $x=x*y$, $y=z$, and $z=\sim y$, then by S 1 and B 6, we have $((x*y)*z)*\sim y=0$. Hence by B 6,

(2)　$(x*y)*z\leqslant\sim y$.

In S 3, put $x=(x*y)*z$, $y=x*z$, and $z=y$, then the right side is equal to 0 by (2) and the second term of left side is to 0 by (1). Therefore we have