24. *Axiom Systems of B-algebra. IV*

By Yoshinari ARAI

(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1966)

In our previous notes (see [1], [2], [3], [4], and [5]), we considered how to formulate axiom systems of propositional calculi into algebraic forms. Among these algebras, we are concerned with the B-algebra which is equivalent to the notion of Boolean algebra. The purpose of our paper is to give some axiom systems of the B-algebra following our new point of view.

In his note (see [4]), K. Iséki defined the B-algebra. Let $\boldsymbol{M}=\langle X, 0, *, \sim\rangle$ be a B-algebra, i.e., $\boldsymbol{M}$ is an abstract algebra which satisfies the following axioms:

$B\,1$ $x*y\leqslant x$,

$B\,2$ $(x*z)*(y*z)\leqslant(x*y)*z$,

$B\,3$ $x*y\leqslant(\sim y)*(\sim x)$,

$B\,4$ $0\leqslant x$,

where $x\leqslant y$ means $x*y=0$, and if $x\leqslant y, y\leqslant x$, then we write $x=y$. As already shown in [1] and [3], the above axiom system is equivalent to the following axioms:

$F\,1$ $x*y\leqslant x$,

$F\,2$ $(x*y)*(z*y)\leqslant(x*z)*y$,

$F\,3$ $(\sim x)*(\sim y)\leqslant y*x$,

$F\,4$ $x\leqslant\sim(\sim x)$,

$F\,5$ $\sim(\sim x)\leqslant x$,

$D\,1$ $0\leqslant x$,

$D\,2$ If $x\leqslant y$ and $y\leqslant x$, then we put $x=y$,

$D\,3$ $x\leqslant y$ means $x*y=0$.

The conditions $F\,1\sim F\,5$ are an algebraic formulation of Frege axioms of classical propositional calculus (see [6]). Therefore a B-algebra is characterized by $F\,1\sim F\,5$ and $D\,1$, $D\,2$, $D\,3$.

In this note, we shall show the following

Theorem. *A B-algebra* $\boldsymbol{M}=\langle X, 0, *, \sim\rangle$ *is characterized by*

$L\,1$ $(x*y)*(x*z)\leqslant z*y$,

$L\,2$ $x\leqslant x*(\sim x)$,

$L\,3$ $x*(\sim y)\leqslant y$,

and $D\,1$, $D\,2$, $D\,3$.

The conditions $L\,1\sim L\,3$ are an algebraic formulation of the two valued classical propositional calculus given by J. Lukasiewicz (see [6]).

First a proof of F$\Rightarrow$L will be given by using the technique in