20. On a Certain Result of Z. Opial

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1. Introduction. In a recent paper [1], Z. Opial proved the following interesting integral inequality:

Theorem. *Let* $y(x)$ *be of class* C' *on* $0\leqslant x\leqslant h$, *and satisfy* $y(0)=y(h)=0$, $y(x)>0$ *on* $(0, h)$. *Then*

$$\int_0^h |yy'|\,dx\leqslant\frac{h}{4}\int_0^h y'^2dx. \tag{1}$$

The constant $h/4$ *is best possible.*

C. Olech [2] showed that (1) is valid for any function which is absolutely continuous on $[0, h]$, and satisfies the boundary conditions $y(0)=y(h)=0$, and Olech's proof of (1) was much simpler than that of Opial. P. R. Beesack [3] gave an even simpler proof of (1) under the hypotheses of Olech, and he also gave more general inequalities of the same type. Later, many simpler proofs were given by N. Levinson [4], C. L. Mallows [5], and R. N. Pederson [6].

By Mallows' method of the proof of (1) we shall give a simple proof of some results of Beesack [3], and show how this method can be used to yield generalization of Opial's and Beesack's inequalities.

2. On the inequality $2\int_a^b |yy'|\,dx\leqslant K\int_a^b py'^2dx$.

Let us define $z(x)=\int_a^x |y'(t)|\,dt$, $a\leqslant x\leqslant X$. Then $|y(x)|\leqslant z(x)$ for $a\leqslant x\leqslant X$, and we have

$$2\int_a^X |y(x)y'(x)|\,dx\leqslant 2\int_a^X zz'dx=z^2(X).$$

Now by the definition of $z(x)$ and Schwarz's inequality

$$z^2(X)=\left(\int_a^X |y'(x)|\,dx\right)^2\leqslant\int_a^X p^{-1}(x)dx\int_a^X py'^2dx.$$

There is equality only if $y=A\int_a^x p^{-1}(t)dt$, A being a constant. Similarly, define $z(x)=-\int_x^b |y'(t)|\,dt$, $X\leqslant x\leqslant b$. Then $|y(x)|\leqslant -z(x)$ for $X\leqslant x\leqslant b$, and

$$2\int_X^b |yy'|\,dx\leqslant 2\int_X^b -zz'dx=z^2(X)=\left(-\int_X^b |y'|\,dx\right)^2\leqslant\int_X^b p^{-1}dx\int_X^b py'^2dx.$$

There is equality only if $y=B\int_x^b p^{-1}(t)dt$, with B constant. Now, we take X such that

$$K=\int_a^X p^{-1}(x)dx=\int_X^b p^{-1}(x)dx, \tag{2}$$