19. A New Convergence Criterion of Fourier Series

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§ 1. The object of this paper is to prove the following two theorems:

Theorem 1. *If* (i) f *is even,* (ii) $\int_0^t f(u)du=o(t)$ *as* $t\to 0$ *and* (iii) *for some* $\delta>0$, *there is an* $\eta(1>\eta>0)$ *such that*

$$\Theta(t)=\int_t^\delta |d\theta(u)|=O(t^{-\eta}) \quad \text{as} \quad t\to 0$$

where $\theta(u)=u^{-\eta}f(u)$, *then the Fourier series of* f *converges at the origin.*

Theorem 2. *If* f *is continuous and is of bounded variation and if there is an* $\eta>0$ *such that* (i) $t^{-\eta}\omega(t)>A>0$ *as* $t\to 0$ *and* (ii)

$$\Theta(t)=\int_t^\delta |d\theta(u)|=O(t^{-\eta}\omega(t)) \quad \text{as} \quad t\to 0$$

uniformly for all x, *where* $\theta(u)=u^{-\eta}\varphi_x(u)$, *then*

$$|s_n(x;f)-f(x)|\leqq A\omega(1/n) \quad \textit{for all} \quad x.$$

§ 2. Proof of Theorem 1. It is sufficient to prove that

$$s_n=\int_0^\delta f(t)\frac{\sin nt}{t}dt=o(1) \quad \text{as} \quad n\to\infty,$$

where δ is a fixed constant. We write $s_n=\int_0^{k/n}+\int_{k/n}^\delta=I_1+I_2$, where k is fixed but a large number. Then $I_1=o(1)$ as $n\to\infty$. By the assumption, $f(u)$ is of bounded variation on the interval $(k/n, \delta)$, and then $|I_2|\leqq V/n$, where V is the total variation of the function $f(t)/t$ on the interval $(k/n, \delta)$. Hence it is sufficient to show that $V=o(n)$. Since $f(t)/t=\theta(t)/t^{1-\eta}$, the required relation is that, for any given ε and a suitable k,

$$\int_{k/n}^\delta \left|d\left(\frac{\theta(t)}{t^{1-\eta}}\right)\right|\leqq \varepsilon n.$$

Now $d\Theta(t)=|d\theta(t)|$ and $|\theta(t)|=\left|\int_t^\delta d\theta(t)\right|\leqq\Theta(t)$ since we can suppose that $f(\delta)=0$ and then

$$\int_{k/n}^\delta \left|d\left(\frac{\theta(t)}{t^{1-\eta}}\right)\right|\leqq\int_{k/n}^\delta\frac{|d\theta(t)|}{t^{1-\eta}}+\int_{k/n}^\delta\frac{|\theta(t)|}{t^{2-\eta}}dt$$

$$\leqq\left[\frac{\Theta(t)}{t^{1-\eta}}\right]_{k/n}^\delta+A\int_{k/n}^\delta\frac{\Theta(t)}{t^{2-\eta}}dt\leqq A\frac{n}{k}<\varepsilon n.$$

This gives the required relation. Thus we get the theorem.