18. An Extension of Certain Quasi-Measure

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(Comm. by Zyoiti SUETUNA, M.J.A., Feb. 12, 1966)

1. Introduction. In 1956, I. M. Gelfand and A. M. Yaglom [3] pointed out importance of probability-theoretical treatments of certain partial differential equations. It would be interesting to construct (signed) measures on function spaces which stand in the same relation to some partial differential equations as the Brownian motion does to the heat equation. Let us consider the Cauchy problem for an equation:

$$\frac{\partial u}{\partial t}=-a\frac{\partial^4 u}{\partial x^4}+b\frac{\partial^2 u}{\partial x^2} \quad (a>0). \tag{1}$$

The solution u with an initial value f is given by

$$u(t, x)=\int_{-\infty}^{\infty} f(y)g(t, x-y)dy,$$

where

$$g(t, x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x-t(a\xi^4+b\xi^2)}d\xi.$$

Let Ω be a function space with a coordinate mapping x_t. It is natural to define a measure $\boldsymbol{P}_x$ of a cylinder set $C=\{\omega : (x_{t_1}(\omega), x_{t_2}(\omega), \cdots, x_{t_n}(\omega))\in \Gamma\}$ in Ω as follows;

$$\boldsymbol{P}_x[C]=\iint\cdots\int_{\Gamma} g(t_1, y_1-x)g(t_2-t_1, y_2-y_1)\cdots \times g(t_n-t_{n-1}, y_n-y_{n-1})dy_1dy_2\cdots dy_n,$$

where $n\geqq 1$, $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n\leqq T$[1] and $\Gamma\subset R^n$. It is easy to see that $\boldsymbol{P}_x$ is well defined on the algebla $\mathfrak{F}$ consisting of all cylinder sets and that it is a finitely additive signed measure on $\mathfrak{F}$. We call $\boldsymbol{P}_x$ a *quasi-measure* corresponding to the equation (1).

Kolmogorov's extension theorem (cf. [5]) shows that if $\boldsymbol{P}_x$ is nonnegative, then $\boldsymbol{P}_x$ has the extension to the σ-algebra $\mathfrak{B}$ generated by $\mathfrak{F}$. But in our case it turns out that $\boldsymbol{P}_x$ may actually be negative and that its total variation is infinite. Therefore $\boldsymbol{P}_x$ can not be extended to $\mathfrak{B}$. This fact was shown in 1960 by V. Yu. Krylov [6] for a wider class of quasi-measures. At the present time we know some sufficient conditions in order that a quasi-measure may be extended to a σ-additive signed measure, which we will call a *Markovian system* (cf. [1], [7]).

In this note we try to obtain a reasonable extension of $\boldsymbol{P}_x$ to

1) Throughout this note, a positive constant T is fixed.