53. *Modular Extensions of Point-Modular Lattices*

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1. **Introduction.** It is well known that a linear space can be represented using properties of the lattice Λ of all subspaces. Properties of certain lattice subsets of Λ are used here to realize this representation. These subsets are designated as point-closed subsystems and a point-modular point-complemented irreducible lattice of length$\geqq 4$ is shown to be a characterization of such a point-closed subsystem. Then a point-modular point-complemented lattice L is decomposed into a subdirect union of point-modular point-complemented irreducible lattices and in the case that L is complete into the direct union. This generalizes the classical representation theory for lattices of the type Λ.

In an application of these results the family of finite dimensional subspaces of a linear space, being a point-closed subsystem of the lattice of all subspaces, is characterized lattice-theoretically. A second application is made to the linear systems of G. W. Mackey [5] for which he states conditions that a family of subspaces of a (real) linear space be the family of closed subspaces relative to some regular linear system constructed on the linear space. Such a family forms a point-closed subsystem and can be used in the description of the linear space. Thus the family of closed subspaces of a regular linear system is characterized lattice-theoretically without the linear space being given explicitly.

The lattice-theoretic notions not defined here or for which no reference is given are in agreement with those of Birkhoff [2]. Let the system $(L, +, \cdot)$ be a lattice. For $S\subset L$ and $b, c\in L$, $(b, c)M_S$ (read (b, c) *modular relative to* S) means $(a+b)c=a+bc$ for every $a\in S$ such that $a\leqq c$. For $a, b\in L$, $a\succ b$ (dually, $b\prec a$) is written for a covers b. The notations $\vee$ and $\wedge$ are set-theoretic union and intersection.

2. **Point-modularity and point-complementation.** In this section $(L, +, \cdot)$ is a lattice with zero 0 and P the set of points in L. The relation M_P is called *point-modularity* and L is said to be point-modular if $M_P=L\times L$. Also L is said to be *point-complemented* if for $a, b\in L$ such that $a<b$ there exists $p\succ 0$ such that $p\leqq b$, $p\nleqq a$. The following brief development is used later.

Lemma 1. *If L is point-complemented and $b, c\in L$ such that*