84. Some Applications of the Functional Representations of Normal Operators in Hilbert Spaces. XX

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Let $T(\lambda)$ be the same notation as that used in the preceding paper; that is, let $T(\lambda)$ be a function with singularities $\overline{\{\lambda_\nu\}} \cup [\bigcup_{j=1}^{n} D_j]$ such that the denumerably infinite set $\{\lambda_\nu\}$ denoting the set of poles of $T(\lambda)$ in the sense of the functional analysis is everywhere dense on a closed or an open rectifiable Jordan curve and that the mutually disjoint closed (connected) domains D_j ($j=1$ to n) have no point in common with the closure $\overline{\{\lambda_\nu\}}$ of $\{\lambda_\nu\}$ and lie in the disc $|\lambda| \leqq \sup_\nu |\lambda_\nu|$.

Theorem 56. Let the ordinary part of such a function $T(\lambda)$ as was stated above be a non-zero constant ξ; let c be an arbitrary finite complex number; let $\sigma = \sup_\nu |\lambda_\nu|$; let $n(\rho, c)$ be the number of c-points, with due count of multiplicity, of $T(\lambda)$ in the closed domain $\bar{\Delta}_\rho\{\lambda: \rho \leqq |\lambda| \leqq +\infty\}$ with $\sigma < \rho < +\infty$; let

$$N(\rho, c) = \int_\rho^{+\infty} \frac{n(r, c) - n(\infty, c)}{r} dr - n(\infty, c) \log \rho \quad (\sigma < \rho < +\infty),$$

$$m(\rho, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} |T(\rho e^{-it})| \, dt \quad (\sigma < \rho < +\infty);$$

and let $M(\rho) = \max_{t \in [0, 2\pi]} |T(\rho e^{-it})|$. Then $\frac{1}{2\pi} \int_0^{2\pi} N(\rho, se^{i\theta}) d\theta$ is a decreasing function of s in the interval $|\xi| < s < M(\rho)$ for every ρ with $\sigma < \rho < +\infty$ and $m(\rho, \infty)$ is a decreasing convex function of $\log \rho$ for the interval $\sigma < \rho < +\infty$; moreover the equality

$$\frac{1}{2\pi} \int_0^{2\pi} N(\rho, se^{i\theta}) d\theta = 0$$

holds for every ρ with $\sigma < \rho < +\infty$ and every s with $M(\rho) \leqq s < +\infty$ and the equation $T(\lambda) - se^{i\theta} = 0$ has no root in the domain $\{\lambda: \rho < |\lambda| < +\infty\}$ for every $\theta \in [0, 2\pi]$ and every s with $M(\rho) \leqq s < +\infty$.

Proof. Consider the function $f(\lambda)$ defined by

$$f(\lambda) = \begin{cases} T\left(\frac{1}{\lambda}\right) = \xi + \sum_{\mu=1}^{\infty} C_{-\mu} \lambda^\mu & (\lambda \neq 0) \\ \xi \quad (\lambda = 0) \end{cases} \quad \left(0 \leqq |\lambda| \leqq \frac{1}{\rho}, \sigma < \rho < +\infty\right),$$

where, as already shown before,

$$C_{-\mu} = \frac{1}{2\pi i} \int_{|\lambda| = \rho'} \frac{T(\lambda)}{\lambda^{-\mu+1}} d\lambda \quad (\sigma < \rho' < +\infty).$$