79. Some Dominated Convergence Theorems in a von Neumann Algebra

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Summary. In his fundamental paper [2], Stinespring proved several dominated convergence theorems for operators measurable *w.r.t.* a gage space. In this paper we state and prove some dominated convergence theorems. One of our theorems is a generalisation of a theorem of Stinespring [2]. In the others, we obtain some results under assumptions, which are weaker than those of Stinespring.

Throughout this paper, the notation and terminology will be the same as those of Segal [1] and Stinespring [2]. Let (H, Ω, m) be a *regular gage* space, namely, H is a complex Hilbert Space, Ω a ring of operators (=von Neumann Algebra) acting on H and m, a gage on Ω, such that for any projection P, $m(P)=0$ implies $P=0$. Denote the L^1 and L^2-space of the gage space by $L^1(H, \Omega, m)=L^1(\Omega, m)$ and $L^2(H, \Omega, m)=L^2(\Omega, m)$, respectively. A sequence of measurable operators (measurable *w.r.t.* Ω in the sense of [1]), is said to converge *grossly* to a measurable operator A, [2, p. 26], if, for every T in $L^1(\Omega, m)$, and for every $\varepsilon>0$, there exists a positive integer N such that for all $n \geqq N$, there exists a projection P_n with the property that $\|(A_n-A)P_n\|<\varepsilon$ and $|m(TQ)|<\varepsilon$ for any projection $Q \precsim I-P_n$ (Q in Ω). Our definition of convergence in measure will be the same as that of Stinespring [2, p. 23]. It is known [2, p. 27] that convergence in measure always implies gross convergence.

Let (A_n) be a sequence of measurable operators converging grossly to a measurable operator A. Suppose there exists an operator B in $L^1(\Omega, m)$, such that $|A_n-A| \leqq B$, $n=1, 2, \cdots$. From a dominated convergence theorem of Stinespring [2, Theorem 4.6] it follows that $A_n-A \rightarrow 0$ in $L^1(\Omega, m)$, hence in particular $m(A_n) \rightarrow m(A)$.

However, there are cases, which are not covered by this Theorem. We give below an example of a sequence (A_n) of non-negative, integrable operators converging grossly to zero (the zero operator) and $m(A_n) \rightarrow m(0)=0$, but there does not exist any integrable operator T with $A_n \leqq T$ for all n. Let Ω be a continuous finite factor and m, the standard faithful normal trace on Ω with $m(I)=1$. Let P_1 be a projection with $m(P_1)=1/2$. Let P_2 be a projection contained in $I-P_1$ with $m(P_2)=1/2^2, \cdots$, and in general P_n a projection contained