78. A Remark on a Theorem of H. Araki

By Katuyosi MATOBA
Kitano Senior High School

(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1966)

1. On the symposium on operator algebras and its applications in physics held at Research Institute for Mathematical Sciences of Kyoto University in the last December, Prof. H. Araki announced, among many others, the following theorem:

Theorem 1. *Let $\mathfrak{A}_i$ be von Neumann algebras acting on a Hilbert space $\mathfrak{H}$ for $i=1, 2, \cdots$. If $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is a factorization and $A_i \in \mathfrak{A}_i$, $\|A_i\| \leqq 1$ for evey i, then*

$$\operatorname*{w-lim}_{i\to\infty} [A_i-(A_i x \mid x)]=0,$$

for an arbitrary $x \in \mathfrak{H}$ with $\|x\|=1$, where w-lim *means the weak operator limit.*

The theorem is a special case of Prof. Araki's more general theorems, cf. [1; § 2, Prop. 4] and [2; § 15, Cor. 2 to Theorem in Remark 5]. However, in the present note, we shall discuss Theorem 1 in the case of the infinite direct product of von Neumann algebras. Since our proof is quite elementary, it may be observed with some interest. Besides, we shall give a sufficient condition that the convergence becomes strong one. Finally, we shall present an example which shows that our theorem fails when we take the complete infinite direct product.

We should like to express our hearty thanks to Prof. H. Araki for his kind guidance.

2. Let $\mathfrak{H}$ be an incomplete infinite direct product of Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2, \cdots$ in the sense of von Neumann [3] and $\mathfrak{R}$ the set of vectors in $\mathfrak{H}$ such that $x=x_1\otimes x_2 \otimes \cdots$ and $\sum_i |\,\|x_i\|-1\,|<+\infty (x_i \in \mathfrak{H}_i$, $i=1, 2, \cdots)$. $\mathfrak{R}$ is a total set in $\mathfrak{H}$. Let A_i be an operator acting on the Hilbert space $\mathfrak{H}_i$ for each $i=1, 2, \cdots$. According to [3; Lemma 6.2.4], each A_i can be considered as an operator acting on $\mathfrak{H}$; especially,

$$A_i x = x_1 \otimes x_2 \otimes \cdots \otimes x_{i-1} \otimes A_i x_i \otimes x_{i+1} \otimes \cdots$$

for $x=x_1 \otimes x_2 \otimes \cdots \in \mathfrak{R}$.

Under these circumstances, we shall prove the following

Theorem 2. *If A_i is an operator on the Hilbert space $\mathfrak{H}_i$ with $\|A_i\| \leqq 1$ for each i, then*

$$\operatorname*{w-lim}_{i\to\infty} [A_i-(A_i x \mid x)]=0, \tag{1}$$

for an arbitrary $x \in \mathfrak{H}$ with $\|x\|=1$.