# 74. On Some Applications of Selberg's Trace Formula

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(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1966)

1. **Definitions.** Let $H$ be the upper half plane and $\Gamma$ be a discrete subgroup of $G=SL(2, \boldsymbol{R})/\{\pm e\}$ acting on $H$ such that $\Gamma\backslash H$ is compact (except in section 5, where $\Gamma$ is the modular group). $d\tau=dxdy/y^2(\tau=x+iy\in H)$ is a $G$-invariant measure on $H$. We will consider an eigenvalue problem of Laplace operator

$$\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

in $L^2(\Gamma\backslash H, d\tau)$. It has non negative discrete spectrum which we denote with $\Lambda$ (or $\Lambda_\Gamma$).

A. Selberg proved the following important trace formula ([2]). If $h(r)$ is an even function satisfying certain analytical conditions, we have

$$(1)\quad \sum_{\lambda=\frac{1}{4}+r^2\in\Lambda} h(r)=\frac{A(D)}{2\pi}\int_{-\infty}^{\infty} r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}h(r)dr+\int_{-\infty}^{\infty}E(r)h(r)dr$$
$$+2\sum_{i=1}^{\infty} n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{\frac{k}{2}}-a_i^{-\frac{k}{2}}}g(k\varepsilon_i),$$

where (i) $A(D)=\int_D d\tau$, $D$ is a fundamental domain of $\Gamma$ in $H$; (ii) $m_\beta(\beta=1, \cdots, s)$ are order of representatives of primitive elliptic classes in $\Gamma$ and

$$E(r)=\frac{1}{2}\sum_{\beta=1}^{s}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\dfrac{k\pi}{m_\beta}}\frac{e^{\pi r-2\pi r\frac{k}{m_\beta}}+e^{-\pi r+2\pi r\frac{k}{m_\beta}}}{e^{\pi r}+e^{-\pi r}};$$

(iii) $1<a_1<a_2<\cdots\uparrow\infty$ are norms, that is square of larger eigenvalues, of representatives $P_\alpha(\alpha=1, 2, \cdots)$ of primitive hyperbolic classes in $\Gamma$, $\varepsilon_i=\log a_i$ and $n_i$ is the number of $P_\alpha$ whose norm is equal to $a_i$; (iv)

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}h(r)dr.$$

In this note we will discuss some applications of this formula: a proof of an asymptotic formula for the eigenvalues (formula (2) in 2), a relation with $\Lambda_\Gamma$ and $\Gamma$ (Theorem in 3), a proof of an anounced result of I. M. Gelfand on a deformation of $\Gamma$ (in 4) and an analogue of formula (2) for the modular group (in 5).