69. Some Properties of $\sum_1^1$- and $\prod_1^1$-sets in N^N

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Introduction. Ljapunow says that there exists a strong connection between the methods of the theory of recursive functions and of the descriptive set theory, and hence the time giving a new expression of the latter theory will present itself ("Einleitung" of [8]). Many authors realize this. Addison calls it the effective descriptive set theory. In this paper, we shall state some results concerning this field. Our main aim is to give the positions in the Kleene analytic hierarchy for sets and ordinals whose existence had been classically proved. When we say a set, it is a subset of N^N so long as we do not mention otherwise, where N^N denotes the set of all 1-place number-theoretic functions and is identified with the Baire zero-space in the usual manner. In our proofs, we shall often use of the following

Effective choice principle. *From every non-empty $\prod_1^1$-set (with respect to the Kleene hierarchy) we can elect a point α_0 such that the unit subset $\{\alpha_0\}$ also belongs to the same class; and hence α_0 itself can be expressed in the Δ_2^1-form (namely, in both the forms $\sum_2^1$ and $\prod_2^1$) as a number-theoretic function* [3; 10; 13].

The method using this principle is suggested by Sampei [11].

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§1. It is well-known that every non-denumerable analytic set (e.g. in the Baire zero-space) contains a non-empty perfect subset. (Cf. e.g., [8].) For any non-denumerable $\sum_1^1$-set E, the following question does arise: In what class a non-empty perfect subset of E can be found? For this, first we can prove the

Theorem 1. *For any recursive R, $|\hat{\beta}(\exists\alpha)(x)R(\bar{\beta}(x), \bar{\alpha}(x))| > \aleph_0$ can be expressed in the $\sum_1^1$-form.*

Using this theorem we obtain the

Theorem 2. *For any non-denumerable $\sum_1^1$-set E there exists a non-empty perfect subset P of E which is O-recursively closed.*[1)]

1) This means that the complement of P is expressible in the form: $CP = \bigcup\{w_n \mid n \in Q\}$ for some Q recursive in O, where O is the Π_1^1-set of natural numbers defined in Kleene [2] and $\{w_n\}_{n=0,1,2,\cdots}$ is a recursive enumeration of all sequence numbers. Here we identify sequence numbers with Baire's intervals in the usual manner.