# 135. *The Cesàro-Perron-Stieltjes Integral. I*

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**1. Introduction.** The Cesàro-Perron integral was introduced by J. C. Burkill [1] by means of major and minor functions using inequalities relating to Cesàro-derivates. In extending such a definition to the Stieltjes type of integration with respect to a general function which may attain the same value at an infinite set, there would be difficulties. We shall define the Cesàro-Perron-Stieltjes integral (*CPS*-integral) by the method of A. J. Ward [5] which uses inequalities concering the increments directly and not in terms of derivates with respect to a function.

The resulting integral is essentially an extension of the Cesàro-Perron integral and we shall prove some continuous and differential properties of the indefinite *CPS*-integral. However the relationship between our integral and the *PS*-integral of A. J. Ward is still open.

**2. Cesàro-continuity and Cesàro-derivates with respect to a function.** Let $f(x), \varphi(x)$ be real valued (finite) functions defined on the interval $[a, b]$. We say that $f(x)$ is *Cesàro-continuous with respect to* $\varphi(x)$ at the point $x_0$, if for some number $K$

$$(1) \qquad \lim_{x\to x_0}\left\{C(f, x_0, x)-f(x_0)-\frac{1}{2}K[\varphi(x)-\varphi(x_0)]\right\}=0,$$

where we put

$$C(f, a, b)=\frac{1}{b-a}\int_a^b f(t)\,dt,$$

the integral being taken in the special Denjoy sense. If in addition we have

$$(2) \qquad \overline{\lim_{x\to x_0+0}}\left\{C(f, x_0, x)-f(x_0)-\frac{1}{2}K[\varphi(x)-\varphi(x_0)]\right\}\Big/\omega(\varphi, [x_0, x])=0$$

then we say that the *right-hand Cesàro-Roussel derivate of* $f(x)$ *with respect to* $\varphi(x)$ at $x_0$ is $K$, where $\omega(\varphi, [x_0, x])$ denotes the oscillation of $\varphi(x)$ on $[x_0, x]$, and write $\overline{CD}_+(f, x_0, \varphi)=K$. The ratio in (2) is to be interpreted to mean 0 whenever its numerator and denominator vanish together. When the oscillation of $\varphi$ is finite, the condition (2) evidently implies (1); however when $\omega(\varphi, [x_0, x])=+\infty$, the condition (1) plays an essential part.

We define three other derivates similarly and put