132. A Fixed Point Theorem for Contraction Mappings in a Uniformly Convex Normed Space

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The purpose of this note is to prove the following

Theorem 1. *Let A be a nonempty, weakly compact and convex subset of a uniformly convex normed space,*[1] *and $\mathcal{F}$ be a nonempty commutative family of contraction mappings*[2] *of A into itself. Then the set of all common fixed points for $\mathcal{F}$ is nonempty, closed and convex.*

This follows from Theorems 2 and 3 below.

Following Brodskii and Milman [1], we say that a bounded convex subset S of a normed space has *normal structure* provided for each convex subset B of S which contains more than one point, there exists a point $a \in B$ such that $\sup_{y \in B} \|a-y\| < d(B)$, where $d(B)$ denotes the diameter of B. A point $a \in B$ is said to be a *diametral point* of B if $\sup_{y \in B} \|a-y\| = d(B)$.

Theorem 2. *Each bounded convex subset of a uniformly convex normed space has normal structure.*

Proof. It is easily seen that in a normed space E if a bounded subset $B \subset E$ which contains more than one point has a nondiametral point $a \in B$, then λa is a nondiametral point of λB for every $\lambda \neq 0$, and $x+a$ is a nondiametral point of $x+B$ for every $x \in E$. Therefore it is sufficient to show that in a uniformly convex normed space, each bounded convex subset B of diameter 1 which has $\{0\}$ as a proper subset, contains a nondiametral point of it.

Assume that 0 is a diametral point of B. Then we can find a sequence $\{a_n\}_{n \geq 2}$ of points of B such that

$$1 \geq \|a_n\| > 1 - \frac{1}{n} \qquad \text{for every } n \geq 2.$$

Suppose that the sequence $\{(1/2)a_n\}_{n \geq 2}$ consists of diametral points of B. Then there exists a sequence $\{b_n\}_{n \geq 2}$ of points of B such that

$$1 \geq \left\| b_n - \frac{1}{2}a_n \right\| > 1 - \frac{1}{n} \qquad \text{for every } n \geq 2,$$

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1) A normed space is said to be *uniformly convex* if $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, and $\lim \|x_n + y_n\| = 2$ imply $\lim \|x_n - y_n\| = 0$.

2) A mapping f of a subset A of a normed space into A is called a *contraction mapping* if $\|f(x) - f(y)\| \leq \|x-y\|$ for all $x, y \in A$.