131. On Smoothness of Normed Lattices

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1. Throughout this note, let $\boldsymbol{R}$ be a continuous semi-ordered linear space (i. e., conditionally σ-complete vector lattice) with a semi-continuous norm, i. e., $\|a_n\| \uparrow_{n=1}^{\infty} \|a\|$ if $0 \leqq a_n \uparrow_{n=1}^{\infty} a$ for a_n and a in $\boldsymbol{R}$.

For mutually orthogonal elements a and b in $\boldsymbol{R}$ with $\|a\|=\|b\|=1$, the curve $C(a, b)$: $\{(\xi, \eta);\ \|\xi a+\eta b\|=1\}$ is called an indicatrix of $\boldsymbol{R}$. $\boldsymbol{R}$ is said to be smooth, if at every point of the unit surface $\boldsymbol{S}$ of $\boldsymbol{R}$ there is only one supporting hyperplane of the unit sphere of $\boldsymbol{R}$, or equivalently the norm on $\boldsymbol{R}$ is differentiable in a sense of Gateaux.

In a normed space, it is well known that the smoothness of the space is equivalent to that of every two dimentional linear subspace. However, in a normed lattice, we shall be able to show that this fact yields from the properties of subspaces spanning by two orthogonal elements in the space.

The purpose of present note is to give a relation between the smoothness of R and the indicatrix of R, which supplement the investigations concerning the indicatrices by H. Nakano [4] and is used for an another material.

Theorem. *In order to $\boldsymbol{R}$ be smooth, it is necessary and sufficient that the norm on is continuous and every indicatrix of $\boldsymbol{R}$ has the tangent line at each point on the curve.*

To prove the theorem, we first state the lemmas.

Lemma 1 (T. Ando [1]). *If $\boldsymbol{R}$ is smooth, then $\boldsymbol{R}$ has a continuous norm.*[1]

x in $\boldsymbol{R}$ is called a *simple element with respect to* a non-zero element a in $\boldsymbol{R}$, if x has a form: $x=\sum_{i=1}^{n}\alpha_i[p_i]$ a where $\{[p_i];\ 1\leqq i\leqq n\}$ is of mutually orthogonal projectors[2] in $\boldsymbol{R}$.

Lemma 2. *When $\boldsymbol{R}$ has a continuous norm, for any $0\neq a\in\boldsymbol{R}$ and $0\leqq x\in\boldsymbol{R}$ there exists a squence of simple elements x_n with respect to a such that*

$$0\text{-}\lim_{n\to\infty} x_n=[a]x. \qquad (0\text{-lim } \textit{means the order limit.})$$

Proof. First let $a>0$. In accordance with [5], let $\mathcal{E}$ be the

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1) The norm $\|\cdot\|$ on $\boldsymbol{R}$ is called a *continuous norm* if $\boldsymbol{R}\ni a_n\downarrow_{n=1}^{\infty}0$ implies $\|a_n\|\downarrow_{n=1}^{\infty}0$.

2) See H. Nakano [5; § 6].