130. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXI

By Sakuji INOUE

Faculty of Science, Kumamoto University

(Comm. by Kinjirô KUNUGI, M.J.A., June 13, 1966)

Definition A. Let $T(\lambda)$ be the function stated in [1]; let $\sigma=\sup_{\nu}|\lambda_{\nu}|$; and let the mutually disjoint, closed, and connected domains D_j $(j=1, 2, 3, \cdots, n)$ which have no point in common with the closure of the denumerably infinite set $\{\lambda_{\nu}\}_{\nu=1,2,3,\ldots}$ be contained in the disc $|\lambda|\leqq\sigma$. Hence, by definition, $T(\lambda)$ is regular in the complex λ-plane $\{\lambda: |\lambda|<+\infty\}$ with the exception of $\overline{\{\lambda_{\nu}\}}\cup\left[\bigcup_{j=1}^{n}D_j\right]$ and every point belonging to the set $\overline{\{\lambda_{\nu}\}}\cup\left[\bigcup_{j=1}^{n}D_j\right]$ is a singularity of $T(\lambda)$. Here $\overline{\{\lambda_{\nu}\}}$ denotes the closure of $\{\lambda_{\nu}\}$.

Theorem 59. Let

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|T(\rho e^{-it})|dt \quad (\sigma<\rho<+\infty).$$

Then

$$\overline{\lim_{\rho\to\sigma+0}}\frac{m(\rho, \infty)}{\log\dfrac{1}{\rho-\sigma}}<+\infty.$$

Proof. Since, as already stated in [1], the sum-function $\chi(\lambda)$ of the first and second principal parts of $T(\lambda)$ is given by

$$\begin{aligned}\chi(\lambda)&=\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}), (f'_{1\alpha}+f'_{2\alpha}))+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\\&=\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_{\alpha}^{(\nu)}}{(\lambda-\lambda_{\nu})^{\alpha}}+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\\&\qquad(1\leqq m, n, k_j<+\infty),\end{aligned}$$

where $\sum_{\nu=1}^{\infty}|c_{\alpha}^{(\nu)}|\leqq\|f_{1\alpha}\|\,\|f'_{1\alpha}\|<+\infty$, we can find from the inequality $\overset{+}{\log}\left|\sum_{\mu=1}^{p}\alpha_{\mu}\right|\leqq\sum_{\mu=1}^{p}\overset{+}{\log}|\alpha_{\mu}|+\log p$ holding for any complex numbers α_{μ} that

$$\begin{aligned}\overset{+}{\log}|T(\rho e^{-it})|\leqq&\overset{+}{\log}|R(\rho e^{-it})|\\&+\overset{+}{\log}\left|\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_{\alpha}^{(\nu)}}{(\rho e^{-it}-\lambda_{\nu})^{\alpha}}\right|+\overset{+}{\log}\left|\sum_{\alpha=1}^{m}((\rho e^{-it}I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})\right|\\&+\overset{+}{\log}\left|\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\rho e^{-it}I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\right|+\log 4 \quad (\sigma<\rho<+\infty),\end{aligned}$$

where $R(\lambda)$ denotes the ordinary part of $T(\lambda)$ and hence is an integral