120. On the Leindler's Theorem

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§ 1. Let f be an integrable function over $(0, 2\pi)$ and periodic with period 2π and let its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We write $\rho_n^2 = a_n^2 + b_n^2$. We shall give a simple proof of the following Leindler theorem (in a little stronger form) [1], [2], [3]:

Theorem 1. a) *If $\lambda(t)$ is a positive decreasing function on $(1, \infty)$, then*

$$(1) \qquad \sum_{n=2}^{\infty} \frac{1}{\lambda(n/2)} \left(\sum_{m=n}^{\infty} \rho_m^q \right)^{1/q} \leq A \int_0^1 \frac{dt}{t^2 \lambda(1/t)} \left(\int_0^{2\pi} | f(x+2t) + f(x-2t) - 2f(x) |^p \, dx \right)^{1/p},$$

where $1 < p \leq 2$ and $1/p + 1/q = 1$.

b) *If $\lambda(t)$ is a positive increasing function on $(1, \infty)$, then* (1) *holds when $\lambda(n/2)$ is replaced by $\lambda(2n)$.*

c) *In the cases* a) *and* b), *the exponents $1/q$ and $1/p$ can be replaced by α/q and α/p where $0 < \alpha \leq q$.*

By our method of proof of Theorem 1, we get another inequality similar to (1) in § 3.

§ 2. **Proof of Theorem 1.** We shall prove only the case a), since the remaining cases may be proved similarly. We have

$$(2) \qquad f(x+2t) + f(x-2t) - 2f(x) \sim 4 \sum_{n=1}^{\infty} \sin^2 nt (a_n \cos nx + b_n \sin nx)$$

and then, by the Hausdorff-Young theorem [4],

$$\left(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} kt \right)^{1/q} \leq A \left(\int_0^{2\pi} | f(x+2t) + f(x-2t) - 2f(x) |^p \, dx \right)^{1/p}.$$

The right side of (1), except for a constant factor, is not less than

$$\int_0^1 \frac{dt}{t^2 \lambda(1/t)} \left(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} kt \right)^{1/q} = \int_1^{\infty} \frac{du}{\lambda(u)} \left(\sum_{k=1}^{\infty} \rho_k^q \sin^{2q} \frac{k}{u} \right)^{1/q}$$

$$\geq \sum_{j=1}^{\infty} \frac{1}{\lambda(2^j)} \int_{2^j}^{2^{j+1}} \left(\sum_{k=2^j}^{\infty} \rho_k^q \sin^{2q} \frac{k}{u} \right)^{1/q} du$$

$$\geq \sum_{j=1}^{\infty} \frac{1}{\lambda(2^j)} \left\{ \sum_{k=2^j}^{\infty} \left(\int_{2^j}^{2^{j+1}} \left(\rho_k^q \sin^{2q} \frac{k}{u} \right)^{1/q} du \right)^q \right\}^{1/q}$$

by the Minkowski inequality. Since it is easy to see that there is