# 167. A Perturbation Theorem for Contraction Semi-Groups

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1. Let $A$ be the infinitesimal generator of a contraction semi-group $\{T(\xi; A); \xi \geqq 0\}$ of class $(C_0)$ on a Banach space $X$. It is well known that

(i) $A$ is a closed linear operator and its domain $D(A)$ is dense in $X$,

(ii) the spectrum of $A$ is located in the half plane $\Re(\lambda) \leqq 0$ and $\| \sigma R(\sigma + i\tau; A) \| \leqq 1$ for $\sigma > 0$, where $R(\sigma + i\tau; A)$ is the resolvent of $A$.

Let $B$ likewise be the infinitesimal generator of another contraction semi-group $\{T(\xi; B); \xi \geqq 0\}$ of class $(C_0)$ on $X$. Recently K. Yosida [7] proved that (i′) if $D(B) \supset D(\hat{A}_\alpha)$, where $\hat{A}_\alpha (0 < \alpha < 1)$ is the fractional power of $A$, then $A + B$ defined on $D(A)$ generates a contraction semi-group of class $(C_0)$, and (ii′) if, moreover, $\{T(\xi; A); \xi \geqq 0\}$ is a holomorphic semi-group, then $A + B$ defined on $D(A)$ generates a holomorphic semi-group.

In this note we shall prove the following theorem.

**Theorem.** *Let $0 < \alpha < 1$ and let $\hat{B}_\alpha$ be the fractional power of $B$.*

(I) *Let us assume that $D(B) \supset D(A)$. Then $A + \hat{B}_\alpha$ defined on $D(A)$ generates a contraction semi-group of class $(C_0)$.*

(II) *Assume, moreover, that $\{T(\xi; A); \xi \geqq 0\}$ is a holomorphic semi-group, then $A + \hat{B}_\alpha$ defined on $D(A)$ also generates a holomorphic semi-group.*

2. Let $B$ be a closed linear operator with domain $D(B)$ and range in a Banach space $X$. Let each positive $\lambda$ belong to the resolvent set of $B$ and let

(1) $$\sup_{\lambda > 0} \| \lambda R(\lambda; B) \| = M < \infty.$$

For $0 < \alpha < 1$, the fractional power $\hat{B}_\alpha = -(-B)^\alpha$ of $B$ is defined as follows:

(2) $$J^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha - 1} R(\lambda; B)(-Bx)\, d\lambda \qquad \text{for } x \in D(B),$$

(3) $\hat{B}_\alpha =$ the smallest closed linear extension of $(-J^\alpha)$.

(See [1], [2], [5], and [6]). Let likewise $A$ be a closed linear operator with domain $D(A)$ and range in $X$ such that "its resolvent