159. An Extension of a Generalized Measure

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1. Introduction. The notion of a topological-additive-group-valued measure has been introduced by the author in [1], which states the process of extending such a measure defined for a certain class of simple sets to one for a wider class of sets. It is noted that any set for which the extended measure is defined is necessarily contained in some set for which the original measure is defined. When, for example, the original measure is given for a class of finite sums of half open intervals in the real line, a measurable set with respect to the extended measure is always bounded.

The purpose of this paper is to extend a topological-additive-group-valued measure to a measure defined for a wider class of sets free of such a restriction.

2. Extension of a measure. Suppose G is a Hausdorff, complete topological additive group with the unit element 0 and ν a G-valued *measure* [1] on a *pseudo-σ-ring* [1] $\mathcal{S}$ of subsets of a fixed set M.

Let Σ be the class of all the pairs $(\mathcal{T}, \lambda)$ such that

1) $\mathcal{T}$ is a pseudo-σ-ring of subsets of M containing $\mathcal{S}$ and each set in $\mathcal{T}$ can be written as a countable sum of sets in $\mathcal{S}$.

2) λ is a G-valued measure defined on $\mathcal{T}$ which is an extension of ν.

Now our purpose is to prove the following theorem.

Theorem 1. *There exists a pair* $(\mathcal{T}_0, \lambda_0)$ *in* Σ *such that* $\mathcal{T}_0 \supset \mathcal{T}$ *and* λ_0 *is an extension of* λ *for any pair* $(\mathcal{T}, \lambda)$ *in* Σ.[1)]

Moreover we have the next theorem when we denote by $\mathcal{L}$ the class of all sets L of the form $L=\bigcup_{i=1}^{\infty} X_i$, $X_i \in \mathcal{S}$, $i=1, 2, \cdots$, having the following property: if $X_i \in \mathcal{S}$, $i=1, 2, \cdots$, $X_i \uparrow L$ as $i \to \infty$ and if $Y_j \in \mathcal{S}$, $j=1, 2, \cdots$, $Y_j \uparrow L$ as $j \to \infty$, then, for any neighbourhood U of 0, there is a positive integer n such that $\nu(X_i) - \nu(Y_j) \in U$ for any $i, j \geqq n$.

Theorem 2. *The pseudo-σ-ring* $\mathcal{T}_0$ *in Theorem* 1 *coincides with the class* $\mathcal{L}$.

We are now going to give a proof of Theorem 1 and Theorem 2.

Denote by $\mathcal{M}$ and $\mathcal{K}$ the class of all the subsets of M and the

1) Obviously such a pair $(\mathcal{T}_0, \lambda_0)$ is unique if it exists.