156. On the Existence of Prime Numbers in Arithmetic Progressions

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The classical theorem of P. G. Lejeune Dirichlet on prime numbers in arithmetic progressions states that, if k and l are two integers with $k\geqq 1$, $(k, l)=1$, then there exist infinitely many primes $p\equiv l \pmod k$. Several elementary proofs are known of this monumental result, with or without the use of the Dirichlet characters to modulus k (cf. e.g. [3; Chap. 9, § 8], [5], [6], [7], [8]), and some of them rest upon the celebrated inequality due to A. Selberg [5]:

$$(1)\qquad \sum_{\substack{p\leqq x\\ p\equiv l \ (\mathrm{mod}\ k)}} \log^2 p+\sum_{\substack{pq\leqq x\\ pq\equiv l \ (\mathrm{mod}\ k)}} \log p\log q=\frac{2}{\phi(k)}x\log x+O(x)$$

as $x\to\infty$, where $\phi(k)$ is the Euler totient function.

Our main interest in the present note is to give another proof of the theorem of Dirichlet on the basis of the inequality (1) by showing that

$$(2)\qquad \sum_{\substack{p\leqq x\\ p\equiv l \ (\mathrm{mod}\ k)}} \frac{\log p}{p}>c\log x \quad (x\to\infty)$$

with some constant $c>0$ (depending on k) implies that

$$\sum_{\substack{p\leqq x\\ p\equiv l \ (\mathrm{mod}\ k)}} \frac{\log p}{p}\sim\frac{1}{\phi(k)}\log x \quad (x\to\infty)$$

for any l relatively prime to k.

It should be noted that we can prove the inequality (2) by an elementary argument (cf. [7, I]). As a matter of fact, a slightly weaker condition than (2) will suffice for our purpose. Indeed, one may replace, on the right-hand side of (2), $c\log x$ by $(\log\log x)^{\alpha}$, α being an arbitrary but fixed real number >1.

1. Let k be a fixed integer $\geqq 1$ and l be any integer with $(k, l)=1$.

We use partial summation to get from (1)

$$(3)\qquad \sum_{\substack{p\leqq x\\ p\equiv l \ (\mathrm{mod}\ k)}} \frac{\log^2 p}{p}+\sum_{\substack{pq\leqq x\\ pq\equiv l \ (\mathrm{mod}\ k)}} \frac{\log p\log q}{pq} =\frac{1}{\phi(k)}\log^2 x+O(\log x).$$

If we put $x=e^n$ and for $(h, k)=1$ $s_m(h)=\sum_{\nu=0}^{m}a_\nu(h)$ $(m=0, 1, 2, \cdots)$, where

$$a_\nu(h)=\sum_{\substack{e^\nu\leqq p<e^{\nu+1}\\ p\equiv h \ (\mathrm{mod}\ k)}} \frac{\log p}{p}=O(1) \quad (\nu=0, 1, 2, \cdots),$$