155. *Algebraic Proof of the Separation Theorem on Dummett's* LC

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Dummett's LC is an intermediate propositional calculus with the following axioms:

1.1 $p \supset q \supset p$.
1.2 $(p \supset q \supset r) \supset (p \supset q) \supset (p \supset r)$.
1.3 $((p \supset q) \supset r) \supset ((q \supset p) \supset r) \supset r$.
1.4 $p \,\&\, q \supset p$ and $p \,\&\, q \supset q$.
1.5 $(p \supset q) \supset (p \supset r) \supset (p \supset q \,\&\, r)$.
1.6 $p \supset p \vee q$ and $q \supset p \vee q$.
1.7 $(p \supset r) \supset (q \supset r) \supset (p \vee q \supset r)$.
1.8 $(p \supset \sim q) \supset (q \supset \sim p)$.
1.9 $\sim p \supset p \supset q$.

The rules of inference are modus ponens and the rule of substitution for variables. We associate to the right and assume the convention that $\supset$ binds less strongly than other connectives. This axiomatization is due to Bull [1].

The separation theorem is the following

Theorem. *A provable formula can be proved by using at most the axioms for implication and those of the connectives which actually appear in the formula.*

The separation theorem on this LC has been proved as a corollary in [4]. But here we show its algebraic proof. The algebraic proof of the theorem was first given by Horn [3] on the intuitionistic system. And we borrow some definitions and results from [3] without mentioning. (So see his paper for those.)

Horn's intuitionistic system is obtained from our system by deleting 1.3. So our definition of an I algebra must differ from his. Our I algebra must satisfy the following conditions:

2.1 If $1 \to x = 1$, then $x = 1$.
2.2 If $x \to y = y \to x = 1$, then $x = y$.
2.3 $x \to y \to x = 1$.
2.4 $(x \to y \to z) \to (x \to y) \to (x \to z) = 1$.
2.5 $((x \to y) \to z) \to ((y \to x) \to z) \to z = 1$.

The last is the one added to Horn's definition.

By Horn the proof of the theorem was reduced to the problem