154. *Semigroups Connected with Equivalence and Congruence Relations*

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0. *Introduction.* The idea of semiclosure operations is useful for finding the smallest equivalence or congruence relation which contains a given relation ρ. It is obtained by applying to ρ the reflexive operation R, symmetric operation S, compatible operation C, and transitive operation T [2], [3]. There are many combinations of operations which give the same equivalence or congruence relation, for example, RST and $RTST$. Using the concept of free semigroup and defining relations we find all words which when interpreted as operations give equivalence and congruence relations; we study the structure of the semigroup generated by R, S, T or R, S, T, C. The defining relations when interpreted as operations are identities. Some results of this paper were published in [2], [3] without proof. Before entering the main discussion we introduce the concept of annexed product or coproduct.

Let G be a groupoid. By G^1 we mean adjoining an identity to G even if G already has one. The annexed product of two groupoids A and B is the direct product of A^1 and B^1 minus the element (1,1).

$$A \tilde{\times} B = A^1 \times B^1 - \{(1, 1)\}.$$

G is isomorphic to $A \tilde{\times} B$ iff G contains two subgroupoids $\hat{A}, \hat{B}$ isomorphic to A and B respectively such that every element of G can be uniquely expressed as a product ab, where $a \in \hat{A}^1$ and $b \in \hat{B}^1$, and the elements of $\hat{A}$ and $\hat{B}$ commute.

1. *Equivalence-Semigroup.* In this section we study the structure of the semigroup generated by R, S, T, [3]. Let Q^* be the semigroup generated by R, S, T, subject to the defining relations (1,1).

(1.1) $R^2=R, S^2=S, T^2=T, RS=SR, RT=TR, STS=TST=ST.$

Theorem 1.1. *Q^* is composed of nine elements.*

(1.2) $R, S, T, RS, RT, ST, TS, RST, RTS.$

Proof. Since R commutes with S and T, if a word contains R, then it has the form $R\cdot W(S, T)$ where $W(S, T)$ is a word of S and T. Let $W(S, T)$ be a word of S and T with length $n \geq 2$. By induction on n we can prove $W(S, T)$ is either ST or TS.

Let I^* be the subsemigroup $\{S, T, ST, TS\}$ of Q^*.