153. *Another Proof of Two Decomposition Theorems of Semigroups*

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(Comm. by Kenjiro SHODA, M.J.A., Sept. 12, 1966)

1. Introduction. One of the early decomposition theorems for semigroups was given by David McLean [2] and may be stated as follows:

Theorem. *An idempotent semigroup S has a greatest semilattice decomposition into rectangular bands.*

In his proof McLean defines a relation σ on S by

$$a\sigma b \text{ if and only if } aba=a \text{ and } bab=b$$

σ is then shown to be the smallest semilattice congruence (abbr. s-congruence) on S. That is, S/σ is a semilattice, and if S/σ' is a semilattice then $\sigma\subseteq\sigma'$. The most difficult part of this proof is in showing the transitivity of σ. We will give another proof based on the concept of "content" of a semigroup and a theorem of T. Tamura [4]. Finally we will give another proof of the following theorem of T. Tamura and N. Kimura [3].

Theorem. *A commutative semigroup S has a greatest semilattice decomposition into archimedean semigroups.*

2. Contents. Definition 1. Let $a_1, a_2, \cdots, a_n$ be elements of a semigroup S. The "content" of $a_1, a_2, \cdots, a_n$ in S, $C_S\langle a_1, a_2, \cdots, a_n\rangle$, is the set of elements of S which can be expressed as a product involving all the elements $a_1, a_2, \cdots, a_n$.

From the definition it is obvious that $C_S\langle a_1, a_2, \cdots, a_n\rangle$ is a subsemigroup of S. As a special case we consider a band.

Lemma 1. *Let S be a band. Then any content $C_S\langle x_1, x_2, \cdots, x_n\rangle$ is a rectangular band.*

To prove Lemma 1 it is sufficient to prove Lemma 2.

Lemma 2. *Let F be a free band generated by $a_1, a_2, \cdots, a_n$. A content $C_F\langle a_1, a_2, \cdots, a_n\rangle$ is a rectangular band.*

However we will prove Lemma 4 which is a more generalized form of Lemma 2.

Let F be the free band generated by $G=\{g_\lambda : \lambda\in\Lambda\}$.

Definition 2. If $X\in F$, let $G(X)=\{g_\lambda\in G\colon X=g_{\lambda 1}g_{\lambda 2}\cdots g_{\lambda n}\}$.[1)]

Lemma 3. *If $X, Y\in F$ then*

(i) $G(XY)=G(X)\cup G(Y)$

1) A similar definition was used by Green and Rees [1].