152. The Lattice of Congruences of Locally Cyclic Semigroups

By Takayuki TAMURA and Wallace ETTERBEEK

University of California, Davis, California

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In [2] Dean and Oehmke proved Theorem 1. Using Theorem 2 proved by Tamura and Levin [4] we will give another proof for Theorem 1.

Theorem 1. *The lattice of congruences on a locally cyclic semigroup is a distributive lattice.*

Theorem 2. *Let S be a locally cyclic semigroup, then $S=\bigcup_{i=1}^{\infty} S_i$ where $S_i \subseteq S_{i+1}$ and S_i is a cyclic semigroup.*

Let C be a cyclic semigroup. Denote C by $C=(n, m)$ where 1 generates C and n, m are non-negative integers or $n=m=\infty$. C is finite if and only if n, m are finite. See p. 19–20 [1].

Any congruence ρ on a cyclic semigroup C is determined uniquely by its induced homomorphic image C' a cyclic semigroup. We denote $\rho=\rho(n', m')$ where $C'=(n', m')$ and

$$(1) \qquad a\rho b \text{ if and only if } \begin{cases} a=b & a<n',\ b<n' \\ m' \mid (a-b) & a\geq n',\ b\geq n'. \end{cases}$$

Proposition 1. *Let $C=(n, m)$ be a cyclic semigroup $\rho=\rho(n_1, m_1)$ is a congruence on C if and only if $n_1 \leq n$, $m_1 \mid m$.*

Proposition 2. *Let* S_1, S_2 be cyclic semigroups such that $S_1 \subseteq S_2$ and 1 generates S_2, k generates S_1. $\rho_1=\rho_1(n_1, m_1)$ and $\rho_2=\rho_2(n_2, m_2)$ are congruences on S_1 and S_2 respectively with $\rho_1=\rho_2 \mid S_1$ if and only if $n_2 \leq n_1$ and $n_1-r \leq n_2-1$ where $n_1 \equiv r (\text{mod}\, k)$, $1 \leq r \leq k$, and $m=\text{lcm}\,(k, m_2)$.

Definition 1. Let σ, ρ be congruences on a groupoid G. Then $\sigma \vee \rho$ is the smallest congruence containing σ and ρ and $\sigma \wedge \rho$ is the largest congruence contained in σ and ρ.

Since the identity relation is contained in all congruences and the universal relation contains all congruences and intersection preserves congruences for any congruences, σ, ρ on a groupoid G both $\sigma \vee \rho$ and $\sigma \wedge \rho$ exist.

In [5] Tamura proved the following.

Proposition 3. *Let C be a cyclic semigroup; let $\sigma=\sigma(n_1, m_1)$, $\rho=\rho(n_2, m_2)$ be congruences on C then*

(i) $\sigma \vee \rho=(\min\,(n_1, n_2), \gcd\,(m_1, m_2))$

(ii) $\sigma \wedge p=(\max\,(n_1, n_2), \text{lcm}\,(m_1, m_2))$.