199. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXIV

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1966)

Theorem 66. For each value of $j=1, 2$, let $\{\lambda_\nu^{(j)}\}_{\nu=1,2,3,\dots}$ be a bounded infinite set of complex numbers; let D_j be a bounded, closed, and connected domain such that the closure $\overline{\{\lambda_\nu^{(j)}\}}$ has not any point in common with it; let N_j be a bounded normal operator whose point spectrum and continuous spectrum are given by $\{\lambda_\nu^{(j)}\}$ and $[\overline{\{\lambda_\nu^{(j)}\}}-\{\lambda_\nu^{(j)}\}]\cup D_j$ respectively (in fact, there exist such $N_j(j=1, 2)$ as we have already demonstrated); let

$$\chi_j(\lambda)=\sum_{\alpha=1}^{m_j}((\lambda I-N_j)^{-\alpha}h_{j\alpha}, g_j) \quad (\lambda \notin \overline{\{\lambda_\nu^{(j)}\}}\cup D_j,\ 1\leqq m_j\leqq\infty,\ j=1, 2),$$

where when $m_j<\infty$ $h_{j\alpha}$ and g_j are arbitrarily given elements in the complex abstract Hilbert space $\mathfrak{H}$ under consideration, whereas when $m_j=\infty$ $\{h_{j\alpha}\}_{\alpha\geqq1}$ are so chosen as to satisfy the condition $\sum_{\alpha=1}^{\infty}\|(\lambda I-N_j)^{-1}\|^\alpha\|h_{j\alpha}\|<\infty$ for any $\lambda\notin\overline{\{\lambda_\nu^{(j)}\}}\cup D_j$ (this is possible); let $U_j(\lambda)=R_j(\lambda)+\chi_j(\lambda)$ where $R_j(\lambda)$ is an integral function; and let Γ be a rectifiable closed Jordan curve containing the sets $\overline{\{\lambda_\nu^{(1)}\}}\cup D_1$ and $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ inside itself. Then

$$(54)\quad \frac{1}{2\pi i}\int_\Gamma U_1(\lambda)U_2(\lambda)d\lambda=\sum_{\alpha=1}^{m_1}\frac{(R_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\frac{(R_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)}{(\alpha-1)!}$$
$$(1\leqq m_j\leqq\infty,\ j=1, 2),$$

the complex line integral along Γ being taken counterclockwise; and moreover the two series on the right both are absolutely convergent when $m_j=\infty(j=1, 2)$. If, in addition to those hypotheses, there exists a rectifiable closed Jordan curve C such that $\overline{\{\lambda_\nu^{(1)}\}}\cup D_1$ lies inside C while $\overline{\{\lambda_\nu^{(2)}\}}\cup D_2$ lies outside C, then

$$(55)\quad \sum_{\alpha=1}^{m_1}\frac{(\chi_2^{(\alpha-1)}(N_1)h_{1\alpha}, g_1)}{(\alpha-1)!}+\sum_{\alpha=1}^{m_2}\frac{(\chi_1^{(\alpha-1)}(N_2)h_{2\alpha}, g_2)}{(\alpha-1)!}=0$$
$$(1\leqq m_j\leqq\infty,\ j=1, 2).$$

Proof. Since

$$\frac{1}{2\pi i}\int_\Gamma R_1(\lambda)R_2(\lambda)d\lambda=0$$

and since, as can be found from the Cauchy theorem and the expansions of $\chi_j\left(\frac{\rho}{\kappa}e^{i\theta}\right)(j=1, 2)$ shown in the preceding papers,