195. On the Convergence of Semi-Groups of Operators

By Shinnosuke ÔHARU

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1. Let X be a locally convex, sequentially complete, linear topological space and $\{U_t^{(n)} : t \geqq 0\}_{n=1,2,\ldots}$ be a sequence of semi-groups of operators on X, satisfying the following conditions:

(i) $U_0^{(n)} = I,\ U_t^{(n)} U_{t'}^{(n)} = U_{t+t'}^{(n)}, \qquad t, t' \geqq 0,$

(ii) $\lim_{t \to t_0} U_t^{(n)} x = U_{t_0}^{(n)} x, \qquad t_0 \geqq 0,\ x \in X,$[1)]

(iii) $\{U_t^{(n)}\}$ are equi-continuous in t and n, i.e., for any continuous semi-norm p on X, there exists a continuous semi-norm q on X, independent of t and n, such that

$$p(U_t^{(n)} x) \leqq q(x) \qquad x \in X.$$

And let $F^{(n)}$ be the infinitesimal generator of $\{U_t^{(n)}\}_{t \geqq 0}$ i.e.,

$$F^{(n)} x = \lim_{h \to 0} h^{-1}(U_h^{(n)} - I)x.$$

We consider the following condition (A):

(A) *There exists a dense linear subset* $\mathfrak{M} \subset \bigcup_{n \geqq 1} \bigcap_{k \geqq n} \mathcal{D}(F^{(k)})$ *such that*

$$\lim_{n,n'} (F^{(n)} x - F^{(n')} x) = 0 \qquad \textit{for each } x \in \mathfrak{M}.$$

M. Hasegawa [2] considered the following problem in the case of Banach space: Under the condition (A), is it true that the additive operator $F = \lim_n F^{(n)}$ or some closed extension of F is the infinitesimal generator of a semi-group $\{U_t\}$ which satisfies $U_t = \lim_n U_t^{(n)}$?

In this paper we shall extend Hasegawa's Theorem on the space X mentioned above and obtain the main theorem:

Theorem 3. *We assume the condition* (A) *and put*

$$Fx = \lim_n F^{(n)} x, \qquad X \in \mathfrak{M}.$$

Then there exists a closed extension $\tilde{F}$ *of the* F *and it generates an equi-continuous semi-group* $\{U_t\}$ *of class* (C_0), *where*

$$U_t x = \lim U_t^{(n)} x, \qquad \textit{for all } x \in X \textit{ and } t \geqq 0,$$

if and only if the following condition (H) *is satisfied:*

(H) *For some* $\lambda_0 > 0$ *and for any continuous semi-norm* p *on* X,

$$\lim_{n,n'} p((I - \lambda_0^{-1} F^{(n)})^{-1} x - (I - \lambda_0^{-1} F^{(n')})^{-1} x) = 0, \qquad x \in X.$$

The proof is given in the section 3.

On the other hand, T. Kato [1] has obtained the following

1) A Semi-group satisfying the conditions (i) and (ii) is said to be of class (C_0).