254. Boolean Multiplicative Closures. II

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In this paper, we shall continue our discussion on Boolean multiplicative closures. The object of this paper is to prove main theorems by using the results of § 2.

3. Boolean multiplicative closures. We recall that the elements $x, y \in L$ are said to be *orthogonal* if $x \wedge y = 0$.

3.1. Lemma. *If ∇ fulfills conditions* C0), C1), *and* C5), *and if x, y are orthogonal elements of L such that $x \wedge y = k \in I(\nabla)$, then $x \in I(\nabla)$ and $y \in I(\nabla)$.*

Proof. By C0), C5), and the orthogonality of x and y we have:

(1) $$0 = \nabla(x \wedge y) = \nabla x \wedge \nabla y.$$

From (1) and C1) we have:

(2) $$y \wedge \nabla x \leq \nabla y \wedge \nabla x = 0.$$

Furthermore, as $x \leq x \vee y = k \in I(\nabla)$, and recalling that C5) implies C3), we have:

(3) $$\nabla x \leq \nabla k = k.$$

Using (2), (3), C3) and the fact that L is distributive, we get:

$$\nabla x = \nabla x \wedge k = \nabla x \wedge (x \vee y) = (\nabla x \wedge x) \vee (\nabla x \wedge y) = x \vee 0 = x,$$

i.e., $x \in I(\nabla)$. Interchanging x and y we have $y \in I(\nabla)$. Q.E.D.

Using the non-distributive lattice with five elements shown in ([2], figure 1, d, page 6) we can see that the distributive condition on L may not be omitted, in general, from 3.1.

We denote by $B = B(L)$ the Boolean algebra of all complemented elements of L. If $b \in B$, $-b$ denotes the complement of b.

An immediate consequence of 3.1 is:

3.2. Theorem. *Let ∇ be as in* 3.1. *Then $B \subset I(\nabla)$.*

A *Boolean multiplicative closure operator* ∇ defined on L is an operator ∇ defined on L such that $\nabla \in \mathrm{Com}(L)$ and $I(\nabla) \subset B(L)$.

We are going to characterize the class £ of all distributive lattices with zero and unit that admits a Boolean multiplicative closure operator.

First of all, we note that according to 3.2., the conditions $\nabla \in \mathrm{Com}(L)$ and $I(\nabla) \subset B(L)$ imply that $I(\nabla) = \nabla(L) = B(L)$. So, if there exists a Boolean multiplicative closure operator ∇ on L it is unique, and moreover, as $B(L)$ is a sublattice of L, $\nabla \in \mathrm{Coam}(L)$ (see 1.1.). Therefore, to solve our problem we must reformulate