249. Note on the Representation of Semi-Groups of Non-Linear Operators

By Shinnosuke ÔHARU

Waseda University

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1. Let X be a Banach space and let $\{T(\xi)\}_{\xi\geqq 0}$ be a family of non-linear operators from X into itself satisfying the following conditions:

(1) $T(0)=I,\ T(\xi)T(\eta)=T(\xi+\eta) \qquad \xi,\eta\geqq 0,$

(2) $\| T(\xi)x-T(\xi)y\|\leqq\| x-y\| \qquad \xi>0,\ x,y\in X,$

(3) There exists a dense subset D in X such that for each $x\in D$, the right derivative

$$D_{\xi}^{+}T(\xi)x=\lim_{h\to 0+}h^{-1}(T(\xi+h)x-T(\xi)x)$$

exists and it is continuous for $\xi\geqq 0$. Then we shall call this family $\{T(\xi)\}_{\xi\geqq 0}$ a *non-linear contraction semi-group*.

Definition. *We define the infinitesimal generator A of a non-linear contraction semi-group $\{T(\xi)\}_{\xi\geqq 0}$ by*

$$Ax=\lim_{h\to 0+}A_h x$$

whenever the limit exists, where $A_h=h^{-1}(T(h)-I)$. We denote the domain of A by $D(A)$.

Lately J. W. Neuberger [1] gave the following result: If $\{T(\xi)\}_{\xi\geqq 0}$ is a non-linear contraction semi-group,*) then for each $x\in X$ and each $\xi\geqq 0$

$$\lim_{n\to\infty}\limsup_{\delta\to 0+}\|(I-(\xi/n)A_\delta)^{-n}x-T(\xi)x\|=0.$$

It is well known that if $\{T(\xi)\}_{\xi\geqq 0}$ is a linear contraction semi-group of class (C_0), then for each $x\in X$ and each $\xi\geqq 0$

$$\lim_{n\to\infty}(I-(\xi/n)A)^{-n}x=T(\xi)x$$

(see [2]). In this paper we shall give the representation of this type for non-linear contraction semi-groups.

The main results are the follwing

Theorem. *Let $\{T(\xi)\}_{\xi\geqq 0}$ be a non-linear contraction semi-group and let A be the infinitesimal generator such that $\overline{\Re(I-\xi_0 A)}=X$ for some $\xi_0>0$. Then for each $\xi>0$ there exists an inverse operator $(I-\xi A)^{-1}$ and its unique extension $L(\xi)$ onto X, which is a contraction operator, and $T(\xi)$ is represented by*

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*) In his paper the following condition is assumed:

(3)′ There is a dense subset D of X such that if x is in D, then the derivative $T'(\xi)x$ is continuous with domain $[0,\infty)$.