241. *Some Theorems on Manifolds of Constant Curvature*

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§1. Riemannian manifolds of constant curvature.

Let M be a connected Riemannian manifold with metric tensor g. We always assume that the dimension n of M is $\geqq 3$. Let ∇ be the covariant differentiation with respect to the Riemannian connection associated with g. The curvature tensor field R is given by

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

where X, Y, and Z are vector fields on M.

Then we have

$$(1)\qquad R(X, Y)+R(Y, X)=0,$$

$$(2)\qquad R(X, Y)Z+R(Y, Z)X+R(Z, X)Y=0 \quad \text{(Bianchi's 1st identity)},$$

$$(3)\qquad (\nabla_X R)(Y, Z)+(\nabla_Y R)(Z, X)+(\nabla_Z R)(X, Y)=0 \quad \text{(Bianchi's 2nd identity)}.$$

The *Riemannian curvature tensor field* of M, denoted also by R, is the tensor field of covariant degree 4 defined by

$$R(X_1, X_2, X_3, X_4)=g(R(X_3, X_4)X_2, X_1).$$

Then R possesses the following properties:

$$(4)\qquad R(X_1, X_2, X_3, X_4)+R(X_2, X_1, X_3, X_4)=0,$$

$$(1')\qquad R(X_1, X_2, X_3, X_4)+R(X_1, X_2, X_4, X_3)=0,$$

$$(5)\qquad R(X_1, X_2, X_3, X_4)=R(X_3, X_4, X_1, X_2),$$

$$(2')\qquad R(X_1, X_2, X_3, X_4)+R(X_1, X_3, X_4, X_2)+R(X_1, X_4, X_2, X_3)=0,$$

$$(3')\qquad (\nabla_{X_5}R)(X_1, X_2, X_3, X_4)+(\nabla_{X_3}R)(X_1, X_2, X_4, X_5)+(\nabla_{X_4}R)(X_1, X_2, X_5, X_3)=0.$$

M is a *Riemannian manifold of constant curvature* if and only if

$$(6)\qquad R(X, Y)Z=k\{g(Y, Z)X-g(X, Z)Y\}$$

where k is a constant.

If R^i_{jkl} and g_{ij} are the components of the curvature tensor field and the metric tensor with respect to a local coordinate system, then the components R_{ijkl} of the Riemannian curvature tensor are given by

$$R_{ijkl}=\sum_{m=1}^{n} g_{im}R^m_{jkl}.$$

If M is a Riemannian manifold of constant curvature, then

$$R^i_{jkl}=k(\delta^i_k g_{jl}-\delta^i_l g_{jk})$$

or

$$R_{ijkl}=k(g_{ik}g_{jl}-g_{il}g_{jk}).$$