240. On Integers Expressible as a Sum of Two Powers

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1. In a recent paper [2] Uchiyama proved the following results.

Theorem 1. *For every $n \geqq 1$ there are positive integers x and y satisfying*

$$n < x^2 + y^2 < n + 2^{\frac{3}{2}} n^{\frac{1}{4}}.$$

Theorem 2. *For any $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^h + y^h < n + (c + \varepsilon) n^a,$$

where

$$a = \left(1 - \frac{1}{h}\right)^2, \quad c = h^{2 - \frac{1}{h}} \quad \text{and} \quad h \geqq 2.$$

The case $h=2$ is due to Bambah and Chowla [1] and Theorem 1 is a refinement of their result.

It is the main purpose of this note to obtain the following refinement of Theorem 2.

Theorem 3. *There is n_0 such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^h + y^h < n + c n^a,$$

where a, c, and h are as in Theorem 2.

This will be deduced from the following result.

Theorem 4. *Let $h \geqq 2$,*

$$N = N(n) = (n^{1/h} + 1)^h - n + 1$$

and

$$g(n) = N - (N^{1/h} - 1)^h.$$

Then for every $n \geqq 1$ there are positive integers x and y satisfying

$$n < x^h + y^h < n + g(n).$$

The case $h=2$ of Theorem 3 is weaker than Theorem 1 which however can be obtained easily from Theorem 4.

Extensions of Theorems 4 and 2 and another result of Uchiyama to sums of the form $x^f + y^h$ (in place of $x^h + y^h$) will be given later.

Our proofs have similarities with those of Uchiyama and Bambah and Chowla.

2. We omit the proof of Theorem 4 which is a special case of Theorem 4A to be proved later.

Theorem 3 follows easily from Theorem 4 and the following